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TECHNICAL PAPERS

AND

DISCUSSIONS

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

PHOTO-ELASTIC DETERMINATION OF SHRINKAGE STRESSES

By Howard G. Smits,[1] Esq.

Synopsis

Stresses due to shrinkage of concrete in masonry dams are of interest and importance, but the determination of these stresses is complex and difficult. In practice, the large masses of concrete used in gravity dams are not homogeneous; nor are they isotropic, as the structure is poured in blocks which present many planes of discontinuity. The indeterminate factors on these planes of discontinuity present such formidable difficulties that an accurate determination of shrinkage stresses is impossible. If the problem is idealized, however, by assuming a homogenous, isotropic, simple gravity dam subject to uniform shrinkage, it becomes possible to predict these stresses with some assurance. Although this idealized case is not truly consistent with actual practice, it creates a philosophical "picture," nevertheless, of the strained condition that should be of value to the judgment of the designing engineer.

One approach to the mathematical solution of this problem has been made by J. H. A. Brahtz,[2] of the United States Bureau of Reclamation. Considering the applicability of the solution of this problem, a complete diagrammatic representation of the stress distribution may be of more interest than the mathematical description. The former is herein presented. This problem was studied at the California Institute of Technology in the photo-elastic laboratory of the Guggenheim School of Aeronautics. The writer wishes to express his appreciation for the continued interest in this problem expressed by Theodor von Kármán, M. Am. Soc. C. E., who is the Director of the Guggenheim School.

Method of Solution

The photo-elastic method is particularly adapted to this problem of determining shrinkage stresses as only a two-dimensional analysis is required.

Note.—Discussion on this paper will be closed in September, 1935, *Proceedings*.

[1] Designing Engr. for Oliver G. Bowen, M. Am. Soc. C. E., Glendale, Calif.

[2] "Stress Distribution in Wedges with Arbitrary Boundary Forces," *Physics*, Vol. 4, No. 2, February, 1933.

A complete description of the theoretical background of the photo-elastic method has been given by Messrs. L. N. G. Filon and E. G. Coker.[3]

The most desirable procedure would be to stretch the dam, *CDE* (Fig. 1), longitudinally, by heat or direct loading; then, in this elongated condition, fasten it securely along the base to the stress free, elastic ground, *AB*. This is a true representation of uniform shrinkage in a theoretical gravity dam section. A line of discontinuity in the stress pattern is encountered along Line *AB*, which is evident from the discontinuity in the elongation on either side of that line. The following general conclusions are evident: (*a*) The crest of the dam, *E*, is free from stress; (*b*) horizontal compression occurs in the ground in the region near the center of the base of the dam; and (*c*) tension gradually dissipates in the regions from Point *G* to Point *A* and from Point *H* to Point *B*.

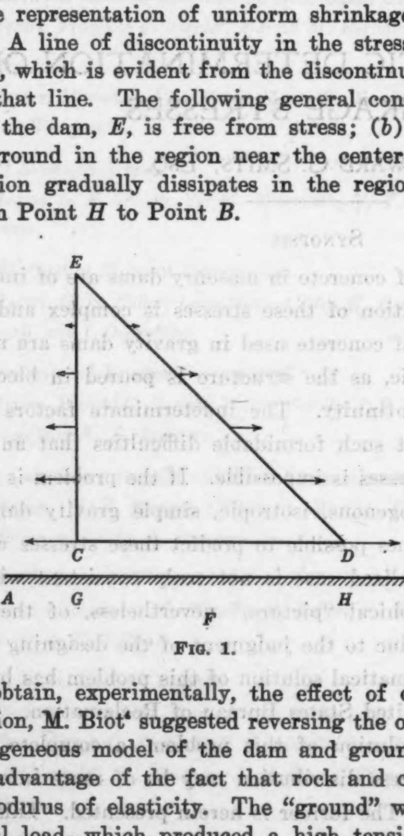

FIG. 1.

FIG. 2.

To obtain, experimentally, the effect of dam shrinkage upon an elastic foundation, M. Biot[4] suggested reversing the operation by elongating the base. A homogeneous model of the dam and ground was made in one piece, thus taking advantage of the fact that rock and concrete have approximately the same modulus of elasticity. The "ground" was then stretched with a direct tensional load, which produced a high tension of, say, p lb per sq in., in Regions *A* and *B* (Fig. 1), which is contrary to the foregoing description of the problem. Since this tension had to be corrected, a blanket horizontal compression of p lb per sq in. was applied to the ground, arbitrarily, after the complete stress analysis had been determined. The result was a stress of zero in the regions, *A* and *B* (Fig. 1); tension in the restricted regions, *G* and *H*; and compression in the region, *F*. A decided discontinuity results along the base line, *CD*. (It is obvious that the addition of a blanket compression is not mathematically exact as (1) the principal tension stress in

[3] "Photo-Elasticity," by E. G. Coker and L. N. G. Filon, Cambridge Press, 1931. (A working summary of the theory, together with some laboratory technique, is given by Mr. Filon in a series of five articles published every two months, in the *General Electric Review*, beginning with the issue for November, 1920.)

[4] "Contribution à la Technique Photo-élastique," *Annales* de la Société Scientifique de Bruxelles, LIII B., 1933.

the ground is not exactly parallel to the ground surface except at infinity, and (2) the ground is partly restrained against movement perpendicular to the ground surface so that some relatively small stress perpendicular to this blanket compression should be added. In this case, however, the error introduced by not considering these two items is within the experimental error and, therefore, their omission is justified.)

The model was made so that the tension could be induced in the ground by means of pins as shown in Fig. 2. To be certain of pure axial loading the tension strap was made symmetrical about its axis; thus, the symmetry of the optical pattern was always a gauge of the degree of axial loading. The loading frame is shown with the model in place in Fig. 3. A load of 1 220 lb per sq in. was used in the strap of the model.

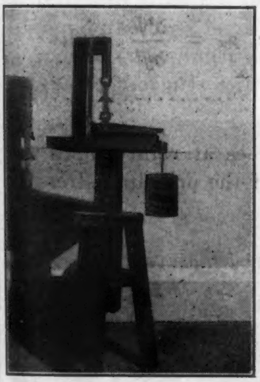

FIG. 3.

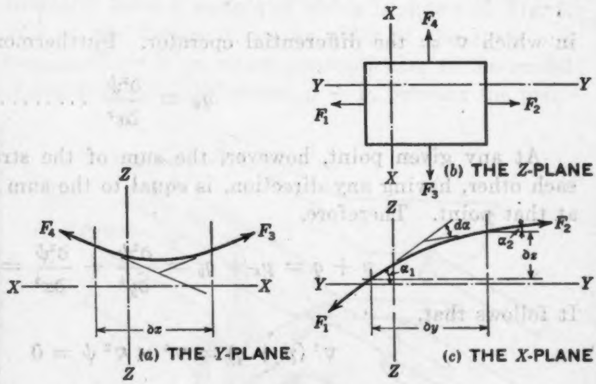

FIG. 4.

The lines of constant shear and the direction lines of principal stress were determined by the standard photo-elastic method. It was desirable furthermore to find the principal stresses. Three methods were open: (*a*) Graphical integration; (*b*) an electrical analogy; and (*c*) a membrane analogy. Graphical integration is the standard method of obtaining the principal stresses, but it is a long and tedious process. No record was found of the electrical analogy having been used in this work. The Prandtl membrane analogy has been used in the solution of Laplace's equation found in problems of shafts carrying torque. At approximately the same time that this problem was solved, E. E. Weibel, at the University of Michigan, was using a soap membrane to find values of $p + q$.[5] Dr. Biot[4] has generalized this analogy theoretically to make it applicable to the photo-elastic method.

THEORY OF NEW METHOD

Desiring a practical and relatively short convenient method of finding the principal stresses, it was decided to test the applicability of the membrane analogy. A rudder membrane was used with success. The analogy is based

[5] See his paper presented at the Annual Meeting of the Am. Soc. Mech. Engrs., December, 1933.

on the fact that both the sum of the principal stresses and the displacement of the membrane obey Laplace's equation. (This statement is true only if the slope of the membrane at every point is small. (See assumption under heading, "The Membrane.") In this experiment the slope of the membrane was excessive only in the restricted regions of the fillets at the toe and heel of the dam. It is in these two regions that the largest experimental errors occurred.)

The Model.—For any given point let p and q be the principal stresses, p_x and q_y, the stresses acting parallel to the respective axes, and ψ, the unknown Airy's stress function. Then,

$$\nabla^4 \psi = 0 = \nabla^2 \times \nabla^2 \psi \quad \text{.......................(1)}$$

in which ∇ = the differential operator. Furthermore, $p_x = \frac{\partial^2 \psi}{\partial y^2}$, and,

$$q_y = \frac{\partial^2 \psi}{\partial x^2} \quad \text{.........................(2)}$$

At any given point, however, the sum of the stresses at right angles to each other, having any direction, is equal to the sum of the principal stresses at that point. Therefore,

$$p + q = p_x + q_y = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = \nabla^2 \psi \quad \text{..............(3)}$$

It follows that,

$$\nabla^2 (p + q) = \nabla^2 \times \nabla^2 \psi = 0 \quad \text{...................(4)}$$

which demonstrates that the sum of the principal stresses satisfies Laplace's equation.

The Membrane.—Let the tension in the membrane be a constant, T, induced by a uniform stretch. Fig. 4 shows any element of the membrane which is in equilibrium, although deformed. (Forces F_1, F_2, F_3, and F_4, are components of the total tension, T; and α = the angle with the horizontal at a given point.) In the X-plane (Fig. 4(c)), the angle that the force makes with the Z-axis is $\tan^{-1} \frac{\partial z}{\partial y}$. If the angle is small, the tangent may be taken equal to the angle; then:

$$F_1 = T \frac{\partial z}{\partial y} \, dy \quad \text{......................(5a)}$$

and,

$$F_2 = T \frac{\partial z}{\partial y} \, dx + T \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) dy \, dx \quad \text{..........(5b)}$$

This gives a resultant component of force parallel to the Z-axis,

$$\frac{\partial}{\partial y} \left(\frac{\partial x}{\partial y} \right) T \, dy \, dx \quad \text{......................(6)}$$

Similar reasoning in the Y-plane leads to,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) T\ dy\ dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which is the component of force parallel to the Z-axis in the Y-plane. To have equilibrium the sum of the two components (Equations (6) and (7)) must equal zero; thus,

$$\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) T\ dy\ dx = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

or, $\nabla^2 z = 0$, from which it is to be noted that z and $(p + q)$ are interchangeable.

The difference, $p - q$, between the principal stresses at any point is determined from the isochromatic lines, a sample of which is shown in Fig. 5. Around the free boundaries of the dam and on the sides of the strap one principal stress must be zero, as there is no stress perpendicular to the model edge. If p or q is zero for any point, the difference, $p - q$, between the principal stresses, and the sum, $p + q$, of the principal stresses, are equal. Along Lines ab and cd in Fig. 2, the total tension is distributed uniformly across the strap so that the sum of the principal stresses is again known. Therefore, the sum, $p + q$, is known completely around the boundaries of the model.

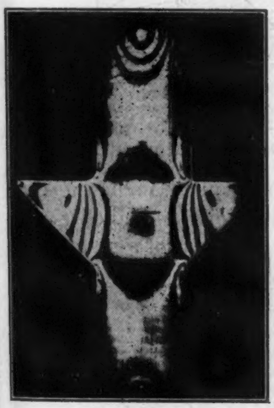

FIG. 5.

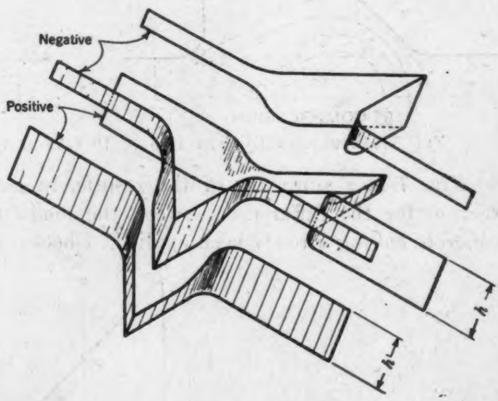

FIG. 6.

APPLICATION OF NEW METHOD

Strips were cut from a thin aluminum plate to form an outline of the model (see Fig. 6). The height, h, of these strips was made proportional at every point to $p + q$ which, as explained previously, was known. Another series of strips was cut to form a negative such that the two fit closely when set one on the other. A rubber sheet, $\frac{1}{64}$ in. thick, was stretched to a uniform tension, T (see Equation (5)), and clamped between the negative and positive dies. A survey of the height of the membrane at every point was made with a micrometer screw on a two-way carriage. This gave a contour map

of the value of $p + q$ for every point. With both $p + q$ and $p - q$ determined (the latter having been found by the usual photo-elastic method), the individual stresses for every point in the model were known.

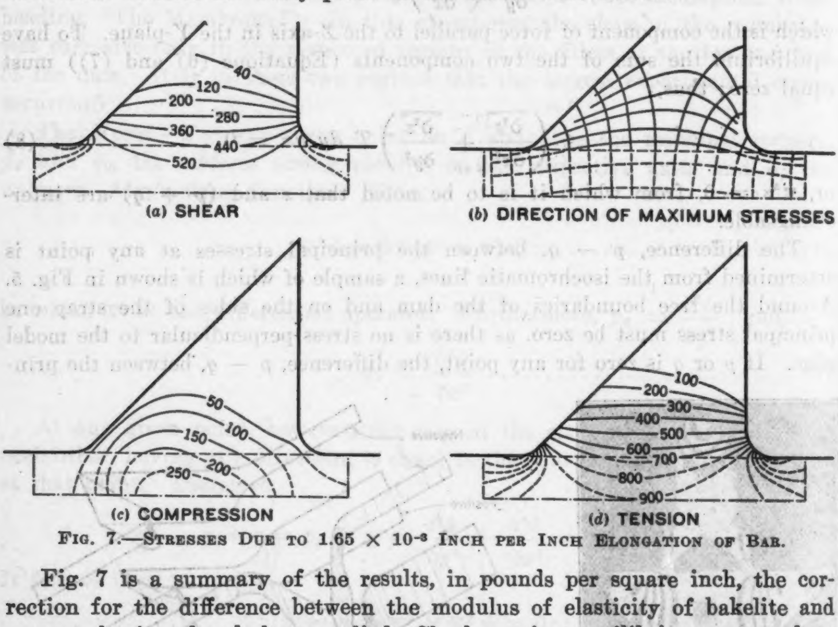

FIG. 7.—STRESSES DUE TO 1.65 × 10^{-3} INCH PER INCH ELONGATION OF BAR.

Fig. 7 is a summary of the results, in pounds per square inch, the correction for the difference between the modulus of elasticity of bakelite and concrete having already been applied. Checks against equilibrium were made to

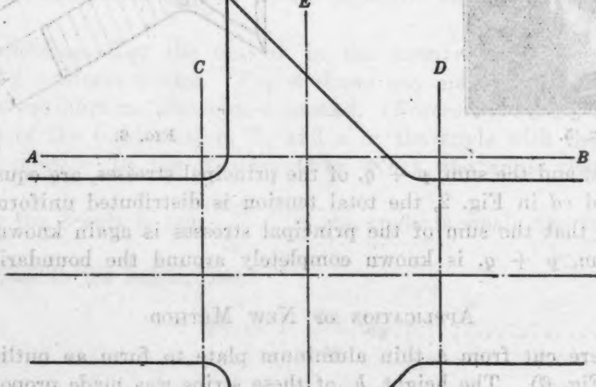

FIG. 8.

determine the experimental accuracy, by obtaining the forces perpendicular to, and the shears parallel to, various sections taken through the model. The results, shown in Figs. 8 and 9, are as follows: Along Section *AB* (Fig. 8),

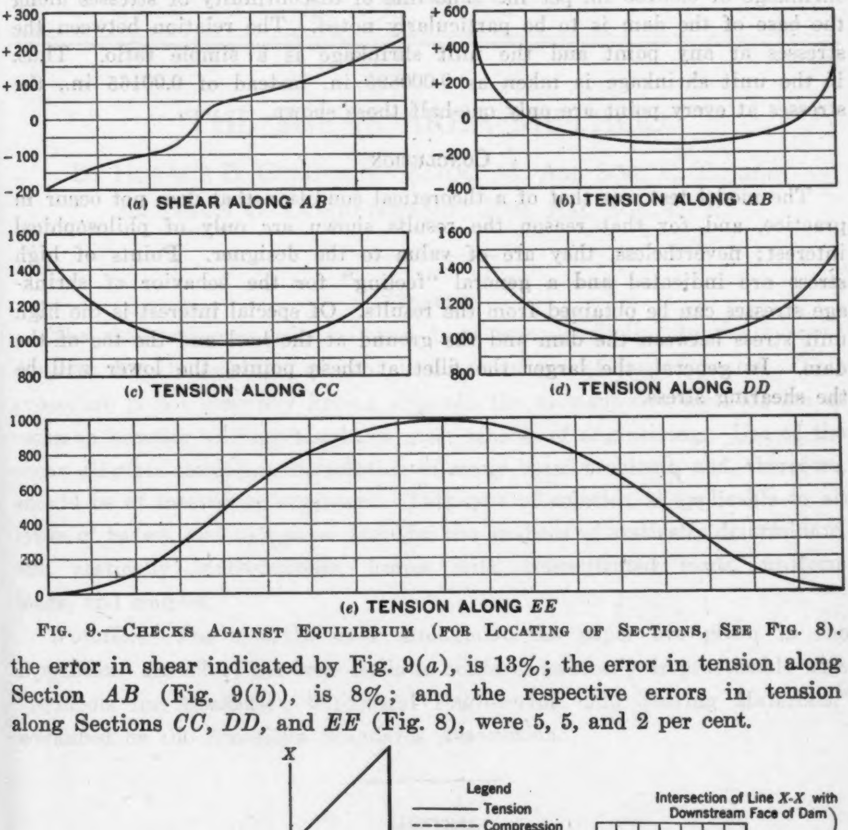

FIG. 9.—CHECKS AGAINST EQUILIBRIUM (FOR LOCATING OF SECTIONS, SEE FIG. 8).

the error in shear indicated by Fig. 9(*a*), is 13%; the error in tension along Section *AB* (Fig. 9(*b*)), is 8%; and the respective errors in tension along Sections *CC*, *DD*, and *EE* (Fig. 8), were 5, 5, and 2 per cent.

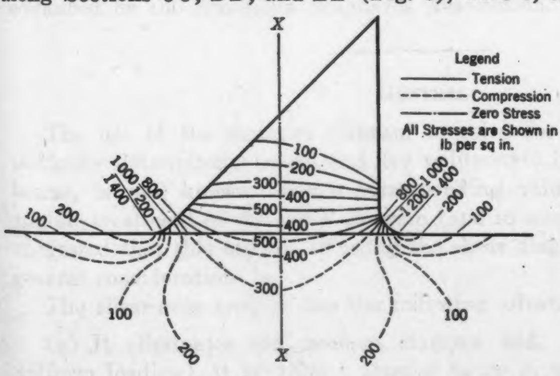

FIG. 10.—STRESSES IN A GRAVITY DAM AND ITS FOUNDATION DUE TO A UNIFORM SHRINKAGE OF 0.0016 INCH PER INCH.

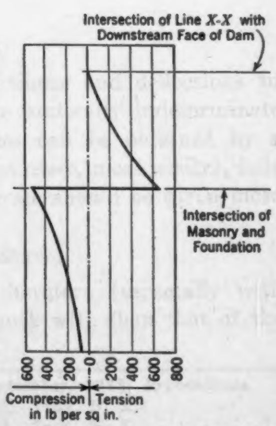

FIG. 11. — DISTRIBUTION OF MAXIMUM TENSION AND COHESION ALONG LINE X-X, FIG. 10.

Finally, a uniform compression is added to that part of the model representing the ground in accordance with the foregoing discussion. Figs. 10 and 11 show the complete stress diagrams for the theoretical condition of a unit shrinkage of 0.00165 in. per in. The line of discontinuity of stresses along the base of the dam is to be particularly noted. The relation between the stresses at any point and the unit shrinkage is a simple ratio. Thus, if the unit shrinkage is taken as 0.000825 in. instead of 0.00165 in., the stresses at every point are only one-half those shown.

Conclusion

The model test was that of a theoretical condition that does not occur in practice, and for that reason the results shown are only of philosophical interest; nevertheless, they are of value to the designer. Points of high stress are indicated and a general "feeling" for the behavior of shrinkage stresses can be obtained from the results. Of special interest is the high unit stress between the dam and the ground at the heel and the toe of the dam. In general, the larger the fillet at these points, the lower will be the shearing stress.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

THE SHEAR-AREA METHOD

By HORACE B. COMPTON,[1] ASSOC. M. AM. SOC. C. E., and CLAYTON O. DOHRENWEND,[2] JUN. AM. SOC. C. E.

SYNOPSIS

The shear diagram may be used in a manner similar to the moment diagram for the solution of the elastic functions of loaded beams. This procedure is not generally known although the moment-area methods are of common practice and are taught in most schools of engineering. Use of the shear diagram simplifies the solution of many beam problems, and, therefore, should be of interest to engineers. This type of solution is applicable to all types of beams, and this paper includes the analysis of statically determinate, and statically indeterminate, beams with concentrated loads, uniform loads, and couples.

***Notation.*—The symbols used throughout the paper are given in the Appendix. An effort has been made to conform as nearly as practicable with "Symbols for Mechanics, Structural Engineering, and Testing Materials," advanced by the American Standards Association.[3]**

GENERAL

The use of the moment diagram to solve for slopes and deflections in statically determinate beams and for redundants in statically indeterminate beams, is well known. Since corresponding values can be obtained by a similar treatment of the shear diagram (and in some cases, more easily), it is suggested that this method of using the shear diagram should be given more general consideration.

The shear-area method has the following advantages:

(*a*) It eliminates the moment diagram and, therefore (especially with uniform loading), it provides a simpler figure to work with than that of the moment-area method;

NOTE.—Discussion on this paper will be closed in September, 1935, *Proceedings*.

[1] Asst. Prof. of Mechanics, Rensselaer Polytechnic Inst., Troy, N. Y.

[2] Instr., Dept. of Civ. Eng., Rensselaer Polytechnic Inst., Troy, N. Y.

[3] A. S. A.—Z10*a*—1932.

(*b*) It simplifies the solution for slope and deflection at any section and affords an easy means of locating the section of maximum deflection; and

(*c*) It provides a ready solution for the derivation of the "Theorem of Three Moments" for any type of loading.

For the student this method will bring into play certain "Mathematical Beams" which will aid his general knowledge of beam action. In the solution of the beam problems the shear diagram is used in the same manner as the moment diagram in the "Conjugate Beam Method."

In considering the shear-area method the following assumptions should be kept clearly in mind:

(1) The modulus of elasticity is considered constant in all the problems;

(2) The shear diagram divided by the moment of inertia is used as the loading on a mathematical beam if the moment of inertia is constant;

(3) The shear at any section of this mathematical beam represents the bending moment at the same section of the real beam;

(4) The bending moment at any section of this mathematical beam represents the slope at the same section of the real beam;

(5) The slope at any section of the mathematical beam represents the deflection at the same section of the real beam (this deflection can be expressed in terms of the moment of inertia of the shear diagram); and,

(6) The end conditions of the mathematical beam are determined by the given beam.

Referring to Assumption (2), when the moment of inertia is not constant the ratio, $\frac{V}{I}$, is replaced by,

$$E\frac{d^3y}{dx^3} = \frac{V}{I} - \frac{M}{I^2}\left(\frac{dI}{dx}\right) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Equation (1) is derived as follows:

$$E\frac{d^2y}{dx^2} = \frac{M}{I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and,

$$E\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{M}{I}\right) = \frac{I\,dM - M\,dI}{I^2\,dx} = \frac{V}{I} - \frac{M}{I^2}\left(\frac{dI}{dx}\right) \ldots\ldots (3)$$

in which, E = modulus of elasticity; y = deflection measured parallel to the Y-axis; x = a distance measured parallel to the X-axis; M = bending moment; I = moment of inertia; and V = shear. Assumptions (1) to (6) are concerned with the relations between beam loading, shear, moment, slope, and deflection. Their significance may be reviewed by reference to Fig. 1 in which:

For shear, Fig. 1(*b*)),

$$V_x = R - \int w_x\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

for moment (Fig. 1(*c*)),

$$M_x = \int V_x\,dx = \int dA \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

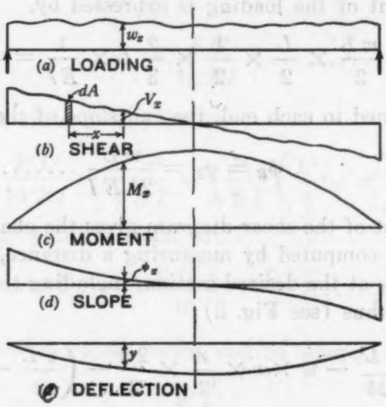

FIG. 1

for slope (Fig. 1(*d*)),

$$\phi_x = \phi_L - \int M_x\, dx = \phi_L - \int x\, dA + C_1 \ldots\ldots\ldots\ldots (6)$$

and, for deflection (Fig. 1(*e*)),

$$y = \int \theta_x\, dx = \theta_L\, x - \int\int x\, dA\, dx = \theta_L\, x - \frac{I_x}{2} + C_2 \ldots\ldots (7)$$

in which R = a reaction; A = the area of a loading diagram on a mathematical beam; and $C_1 = C_2$ = a constant (= 0).

Analysis of Statically Determinate Beams

The following typical cases are given as a development of the shear-area method of solving problems applied to beams that are statically determinate.

Case 1.—Simple Beam with Uniform Load Over Entire Span.—In Case 1, the real beam (Fig. 2(*a*)) has slope at each end and, therefore, the mathe-

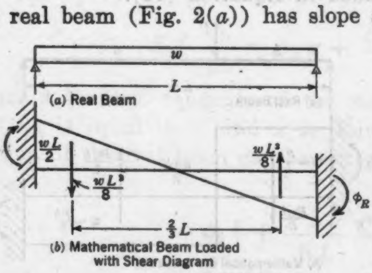

FIG. 2

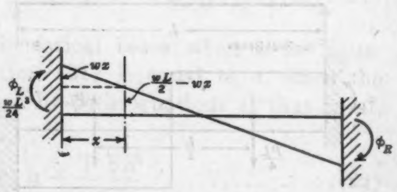

FIG. 3

matical beam (Fig. 2(*b*)) has a moment at each end. The real beam has no bending moment at the ends and, therefore, the mathematical beam has no shear at the ends. Hence, the reactions of this mathematical beam are couples. The loading is also a couple and is placed symmetrically about the center and, consequently, one-half the total couple is carried by each end reaction. This is evident since in this case the real beam has equal end slopes.

The total moment of the loading is expressed by,

$$\theta_t = \frac{wL}{2} \times \frac{L}{2} \times \frac{1}{2} \times \frac{2}{3} L \times \frac{1}{EI} = \frac{wL^3}{12\,EI} \quad \text{..........(8)}$$

With one-half assigned to each end, the end slope of the real beam is:

$$\phi_R = \phi_L = \frac{wL^3}{24\,EI} \quad \text{.......................(9)}$$

Since the moment of the shear diagram gives the change in slope, the slope at any point can be computed by measuring a distance, x, from one end and then taking moments at the desired section, including the end reactions in the moment equations; thus (see Fig. 3):

$$\phi_x = \frac{wL^3}{24} - w \times x \times \frac{x}{2} \times \frac{2x}{3} - \left(\frac{wL}{2} - wx\right)\frac{x^2}{2}$$

$$= \left(\frac{wL^3}{24} + \frac{wx^3}{6} - \frac{wLx^2}{4}\right)\frac{1}{EI} \quad \text{...............(10)}$$

To find the deflection, y, at any section integrate Equation (10):

$$EI\,y = \int_0^x \frac{wL^3\,dx}{24} + \int_0^x \frac{wx^3\,dx}{6} - \int_0^x \frac{wLx^2\,dx}{4}$$

$$= \frac{wL^3x}{24} - \left(\frac{wLx^3}{12} - \frac{wx^4}{24}\right) = \frac{wL^3x}{24} - \frac{I_x}{2} \quad \text{........(11)}$$

The maximum deflection occurs at the section where $x = \frac{L}{2}$:

$$y_m = \frac{5}{384}\frac{wL^4}{EI} \quad \text{.......................(12)}$$

When concentrations occur in the shear diagram each value is multiplied by the square of its distance and included in Equation (11).

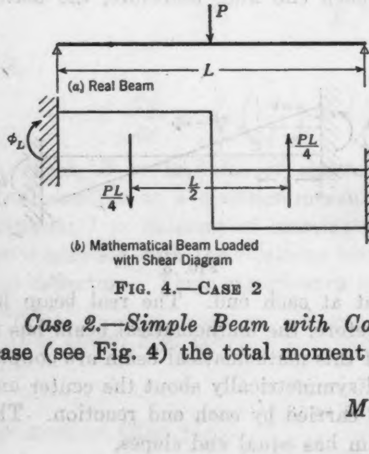

FIG. 4.—CASE 2

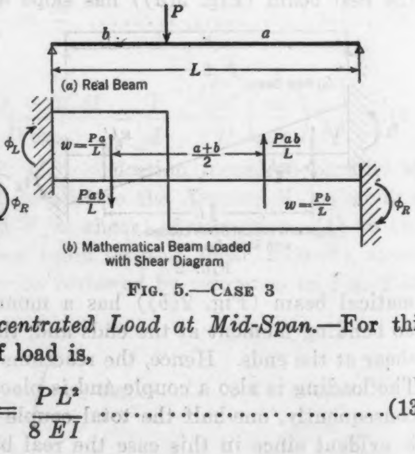

FIG. 5.—CASE 3

Case 2.—Simple Beam with Concentrated Load at Mid-Span.—For this case (see Fig. 4) the total moment of load is,

$$M_T = \frac{PL^2}{8\,EI} \quad \text{...........................(13)}$$

and the end slope is,

$$\phi_L = \frac{P L^2}{16 E I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Taking moments for slope at any point:

$$\phi_x = \frac{P L^2}{16 E I} - \frac{P x^2}{4 E I} = \frac{P}{4 E I}\left(\frac{L^2}{4} - x^2\right) \ldots\ldots\ldots\ldots (15)$$

The deflection at any point is equal to:

$$E I y = \frac{P L^2 x}{16} - \frac{I_x}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

and the maximum deflection occurs at the section where $x = \frac{L}{2}$:

$$y_m = \frac{P L^3}{48 E I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Case 3.—Simple Beam with Concentrated Load Not at Mid-Span.—To determine the end conditions of the mathematical beam for this case (see Fig. 5), write the general moment equation with the left end as the origin, thus,

$$E I \frac{d^2y}{dx^2} = \phi_L - \frac{P a x^2}{2 L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

and,

$$E I \frac{dy}{dx} = \phi_L x - \frac{P a x^3}{6 L} + C\ (= 0) \ldots\ldots\ldots\ldots\ldots (19)$$

The moment equation as derived from the right end is,

$$E I \frac{d^2y}{dx^2} = \phi_R + \frac{P b x^2}{2 L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

or,

$$E I \frac{dy}{dx} = \phi_R x + \frac{P b x^3}{6 L} + C\ (= 0) \ldots\ldots\ldots\ldots\ldots (21)$$

Equate the two tangents of the mathematical beam when x in Equation (19) is equal to b, and x in Equation (21) is equal to a, since the deflection of the real beam can be expressed by both equations at that point, thus:

$$- \phi_L b + \frac{P a b^3}{6 L} = \phi_R a + \frac{P b a^3}{6 L} \ldots\ldots\ldots\ldots\ldots (22)$$

Furthermore, since $\phi_R = \phi_L - \phi_T$:

$$\phi_L = \frac{P a b}{6 L}(L + a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

and,

$$\phi_R = - \frac{P a b}{6 L}(L + b) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

The equation for deflection and slope for any section can now be readily written: For slope,

$$\phi_x = \frac{P\,a\,b}{6\,L}(L + a) - \frac{P\,a\,x^2}{2\,L} \qquad (25)$$

and for deflection,

$$y = \phi_L x - \frac{I_x}{2} \qquad (26)$$

Since at the section of maximum deflection the slope is equal to zero in the real beam, the moment equation of the mathematical beam need only be set to zero and solved for the value of x; thus:

$$-\frac{P\,a\,b}{6\,L}(L + b) + \frac{P\,b\,x^2}{2\,L} = 0 \qquad (27)$$

or,

$$x = \sqrt{\frac{a}{3}(L + b)} \qquad (28)$$

Substituting Equation (28) in the deflection equation,

$$y_m = \frac{P\,a\,b}{9\,E\,I\,L}\sqrt{\frac{a}{3}(L + b)^3} \qquad (29)$$

It is interesting to note that, as the value of b changes from zero to $\frac{1}{2} L$, the value of x varies from $\frac{1}{\sqrt{3}} L$ to $\frac{1}{2} L$. This means that x_m will always be such as to bring the section of maximum deflection within 8% of L, distant from the middle of the beam.

Case 4.—Simple Beam with Segment of Uniform Load at Any Position.—Equations (23) and (24) may be used to determine the end condition of the

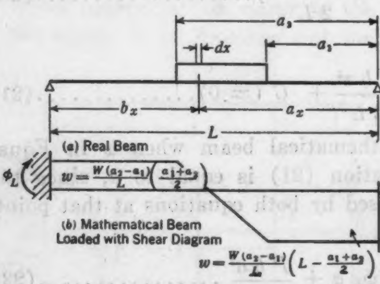

FIG. 6.—CASES 4 AND 5.

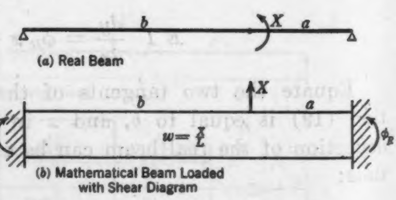

FIG. 7.—CASE 5.

mathematical beam in the case of a simple beam with a segment of uniform load at any position (see Fig. 6). Then (see Fig. 6(a)),

$$\phi_L = \int_{a_1}^{a_2} \frac{w\,dx \times a_x\,(L - a_x)}{6\,L}(L + a_x)$$

$$= \frac{w}{6\,L}\left[\frac{L^2}{2}(a_2^2 - a_1^2) - \frac{1}{4}(a_2^4 - a_1^4)\right] \qquad (30)$$

The slope at any point can now be obtained from the moment equation of the mathematical beam; thus, for values of x from zero to $(L - a_2)$ with the origin at the left support,

$$\phi_x = \phi_L - w\,(a_2 - a_1)\left(\frac{a_1}{2} + \frac{a_2}{2}\right)\frac{x^2}{2\,L} \quad \ldots\ldots\ldots\ldots\ldots (31)$$

The deflection at any point can be expressed in the same manner as in Case 4 (see Equation (26)).

Case 5.—Simple Beam with Couple Applied at Any Section.—Considering the real beam first, it will be noted that at the section where the couple is applied there is an abrupt change in the bending moment. Since the shear of the shear diagram is to represent the bending moment of the real beam, this means that a concentrated load must be placed on the shear diagram at that point equal to the magnitude of the couple (see Fig. 7).

The end conditions of the mathematical beam are determined as in Case 3. Taking moments from the left end (see Fig. 7): $E\,I\,\frac{d^2y}{dx^2} = \phi_L - \frac{X\,x^2}{2\,L}$; and,

$$E\,I\,\frac{dy}{dx} = \phi_L\,x - \frac{X\,x^3}{6\,L} \quad \ldots\ldots\ldots\ldots\ldots (32a)$$

in which, X, Fig. 7, is the couple in question. Taking moments from the right end, $E\,I\,\frac{d^2y}{dx^2} = \phi_R + \frac{X\,x^2}{2\,L}$; and,

$$E\,I\,\frac{dy}{dx} = \phi_R\,x + \frac{X\,x^3}{6\,L} \quad \ldots\ldots\ldots\ldots\ldots (32b)$$

Equate tangents when $x = b$ in Equation (32a), and $x = a$ in Equation (32b), remembering that the sum of the reaction couples must equal the couple due to the loading. In other words, $\phi_L + \phi_R = \frac{X}{L} \times L\left(\frac{L}{2} - a\right)$; or, $\phi_L + \phi_R - \frac{X\,L^2}{L\,2} + Xa = 0$. Then, re-introducing E and I:

$$\phi_L = \left(\frac{X\,b^3}{6\,L^2} + \frac{X\,a\,b^2}{2\,L^2} - \frac{X\,a^3}{3\,L^2}\right)\frac{1}{E\,I} \quad \ldots\ldots\ldots\ldots (33a)$$

and,

$$\phi_R = \left(-\frac{X\,a^3}{6\,L^2} - \frac{X\,b\,a^2}{2\,L^2} + \frac{X\,b^3}{3\,L^2}\right)\frac{1}{E\,I} \quad \ldots\ldots\ldots\ldots (33b)$$

The slope and deflection at any point are quickly computed, as before (in b segment):

$$\phi_x = \phi_L - \frac{X\,x^2}{2\,L} \quad \ldots\ldots\ldots\ldots\ldots (34)$$

and,

$$E\,I\,y = \phi_L\,x - \frac{I_x}{2} \quad \ldots\ldots\ldots\ldots\ldots (35)$$

Case 6.—Cantilever Beam with Concentrated Load at the End.—For Case 6, refer to Fig. 8:

$$\phi_A = \frac{M L}{E I} - \frac{P L^2}{2 E I} = \frac{P L^2}{2 E I} \quad \text{.................. (36)}$$

$$\phi_y = \frac{M x}{E I} - \frac{P x^2}{2 E I} \quad \text{...................... (37)}$$

and,

$$E I y = \frac{I_x}{2} = \frac{M x^2}{2} - \frac{P x^3}{6} \quad \text{.................. (38)}$$

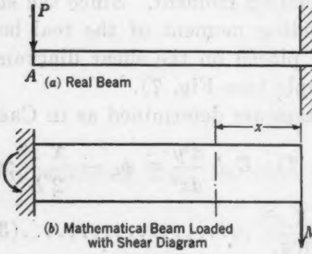

FIG. 8

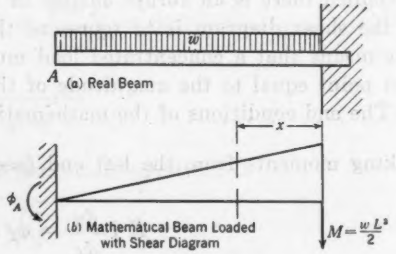

FIG. 9

The maximum deflection occurs when $x = L$, or,

$$y_m = \frac{P L^3}{3 E I} \quad \text{...................... (39)}$$

Case 7.—Cantilever Beam with Uniform Load Over Entire Span.—For this case, (see Fig. 9):

$$\phi_A = \frac{M L}{E I} - \frac{w L^3}{2 E I} \times \frac{2}{3} L = \frac{w L^3}{6 E I} \quad \text{.............. (40a)}$$

$$\phi_x = \frac{w L^2 x}{2 E I} - \frac{w (L - x)}{E I} \frac{x^2}{2} - w \times x \times \frac{x}{2} \times \frac{2 x}{3} \quad \text{...... (40b)}$$

and, $E I y = \frac{I_x}{2}$ (see Equation (38)) $= \frac{M x^2}{2} - \frac{w x^4}{8} - \frac{(w L - w x)}{6} x^3$. The maximum deflection occurs where $x = L$, or,

$$y_m = \frac{w L^4}{8 E I} \quad \text{...................... (41)}$$

Case 8.—Simple Beam with Variable Moment of Inertia and with Concentrated Load at Mid-Span.—The concentrated load in Fig. 10 is due to the abrupt change in the moment of inertia which makes the shear curve of the mathematical beam drop directly at that point. Its value is determined by the extra area added, due to the difference in I. Since the mathematical beam is loaded symmetrically one-half the total loading moment will go to each end, or,

$$\phi_L = \phi_R = \frac{P}{2} \times \frac{L}{2} \times \frac{L}{4} + \frac{P}{2} \times \frac{L}{4} \times \frac{3}{8} L - \left(\frac{P L}{8}\right) \times \frac{L}{4} = \frac{5}{64} \frac{P L^2}{E I} \quad \text{. (42)}$$

The maximum deflection can be found in the usual manner (see Equation (35)). It occurs at the section where $x = \frac{L}{2}$, or,

$$y_m = \frac{3}{128}\,\frac{P\,L^3}{E\,I} \quad \text{.......................} (43)$$

Example in Statically Determinate Beam

As an example, consider the loading in Fig. 11. By Equation (23),

$$\phi_A = \sum \frac{P\,a\,b}{6\,L}(L + a) = \frac{2000 \times 8 \times 10 \times 26 + 1000 \times 14 \times 4 \times 32}{108} = \frac{55\,111}{E\,I}.$$

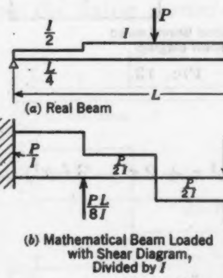

Fig. 10

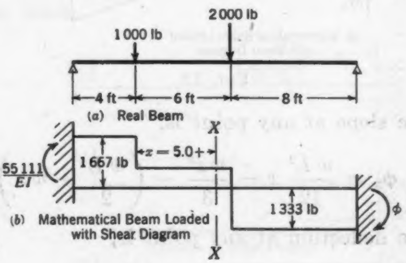

Fig. 11

The moment of the mathematical beam being equal to zero at the section of maximum deflection: $M = 0 = 55\,111 - 1\,667 \times 4\,(2 + x) - 667\,\frac{x^2}{2}$; and $x = 5.0+$ (use $x = 5$).

By Equation (26),

$$E\,I\,y = \phi_a x - \frac{1}{2}\,I_x$$

$$= 55\,111 \times 9 - \frac{1}{2}\left(1\,667 \times \frac{4^3}{12} + 1\,667 \times 4 \times 7^2.0 + 667 \times \frac{5^3}{3}\right)$$

and

$$y_m = \frac{314\,300}{E\,I}.$$

Analysis of Statically Indeterminate Beams

The following typical problems are given as a development of the shear-area method of solving problems applied to beams that are statically indeterminate.

Case 9.—Fixed Beam with Uniform Load Over Entire Span.—The mathematical beam for this case (see Fig. 12) has a symmetrical load and is held in equilibrium by end shears, each of which is the end moment of the real beam. In other words,

$$M_I = \frac{w\,L}{2} \times \frac{L}{2} \times \frac{1}{2} \times \frac{2\,L}{3} \times \frac{1}{L} = \frac{w\,L^2}{12} \quad \text{..........} (44)$$

The moment at the center of the beam is,

$$M_m = \frac{w L^2}{12} - \frac{w L^2}{8} = -\frac{w L^2}{24} \quad \text{.................(45)}$$

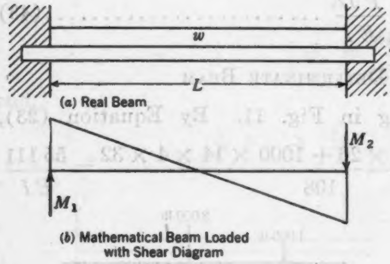

FIG. 12

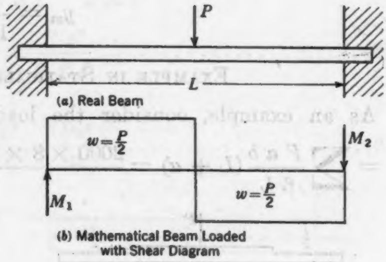

FIG. 13

the slope at any point is,

$$\phi_x = \frac{w L^2}{12} x - \frac{w x^3}{3} - \left(\frac{w L}{2} - w x\right) \frac{x^2}{2} = \frac{w}{12} (L^2 x + 2 x^3 - 3 L x^2) \quad \text{..(46)}$$

the deflection at any point is,

$$E y = \frac{I_x}{2} \quad \text{.........................(47)}$$

and the deflection is a maximum at the section where $x = \frac{L}{2}$, or,

$$y_m = \frac{1}{2} \left[\frac{w L^2}{12} \left(\frac{L}{2}\right)^2 - \frac{w L}{2} \left(\frac{L^3}{2}\right) \frac{1}{4} \right] = \frac{w L^4}{384 E I} \quad \text{......(48)}$$

The points of inflection will occur where the shear of the mathematical beam is equal to zero; that is, $\frac{w L^2}{12} - \frac{w x^2}{2} - \frac{w L x}{2} + w x^2 = 0$; or, $x = 0.211 L$, or $0.789 L$.

Case 10.—Fixed Beam with Concentrated Load at Mid-Span.—For this case (see Fig. 13), the end moment is:

$$M_1 = \frac{P L^2}{8 L} = \frac{P L}{8} \quad \text{......................(49)}$$

the slope at any point is,

$$\phi_x = \frac{P L x}{8} - \frac{P x^2}{4} \quad \text{..................(50)}$$

the deflection at any point is equal to $E I y = \frac{I_x}{2}$; and this deflection is a maximum at the section where $x = \frac{L}{2}$; that is,

$$y_m = \frac{1}{2} \left[\frac{P L}{8} \left(\frac{L}{2}\right)^2 - \frac{P}{2} \left(\frac{L}{2}\right)^3 \frac{1}{3} \right] = \frac{P L^3}{192 E I} \quad \text{.......(51)}$$

A point of inflection of the real beam will occur at the section (see Fig. 13) where the shear of the mathematical beam is zero. Consequently, $\frac{P L}{8} - \frac{P x}{2} = 0$; and, $x = \frac{L}{4}$.

Case 11.—Fixed Beam with a Concentrated Load Not at the Mid-Span.—Consider the shear diagram in two parts; the first (Fig. 14(*b*)) as if the span were simply supported, and the second (Fig. 14(*c*)) with constant shear produced by the fixing moments. Since the slope at each support due to the combined effect is zero, one may solve end slopes due to a general beam carrying end moments only and equate to the end slopes of the same beam as a simply supported one carrying the load. This method then permits solving the fixing moments of the real beam.

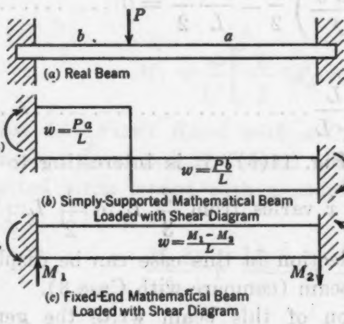

FIG. 14

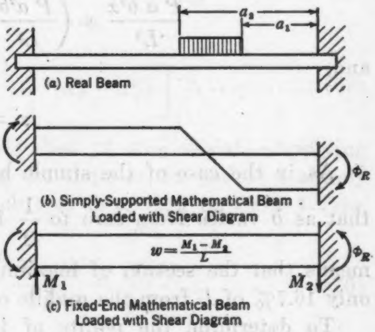

FIG. 15

A derivation of the end moments of a mathematical beam loaded with constant shear (see Fig. 14(*c*)) is as follows: The moment of the mathematical beam is expressed by,

$$M = \phi_L - M_1 x + \left(\frac{M_1 - M_2}{L}\right)\frac{x^2}{2} \quad \text{.............(52)}$$

the tangent of the mathematical beam is:

$$\phi_x = \phi_L x - M_1 \frac{x^2}{2} + \left(\frac{M_1 - M_2}{L}\right)\frac{x^3}{3} \quad \text{.............(53)}$$

Since Equation (53) equals zero when $x = L$:

$$\phi_L = \frac{M_1 L}{3} + \frac{M_2 L}{6} \quad \text{.......................(54)}$$

and,

$$\phi_R = \frac{M_1 L}{6} + \frac{M_2 L}{3} \quad \text{.......................(55)}$$

The combined end slope of the real beam is zero; therefore:

$$\frac{M_1 L}{3} + \frac{M_2 L}{6} = \frac{P\,a\,b}{6\,L}(L + a) \quad \text{.................(56)}$$

$$\frac{M_1 L}{6} + \frac{M_2 L}{3} = \frac{P\,a\,b}{6\,L}(L + b) \quad \text{.................(57)}$$

and, therefore,

$$M_1 = \frac{P\,b\,a^2}{L^2} \quad\text{.........................(58)}$$

and,

$$M_2 = \frac{P a b^2}{L^2} \quad\text{.........................(59)}$$

To determine the section of maximum deflection equate the moment equation of the complete mathematical beam to zero and solve for x, thus:

$$M_2 x + \left(\frac{M_1 - M_2}{L}\right)\frac{x^2}{2} - \left(\frac{Pb}{L}\right)\frac{x^2}{2} = 0 \quad\text{.............(60)}$$

$$\frac{P\,a\,b^2 x}{L^2} + \left(\frac{P\,a^2 b}{L^2} - \frac{P\,a\,b^2}{L^2}\right)\frac{x^2}{2} - \frac{P\,b}{L}\frac{x^2}{2} = 0 \quad\text{..........(61)}$$

and,

$$x = \frac{2\,a\,L}{2\,a + L} \quad\text{.........................(62)}$$

As in the case of the simple beam (Fig. 14(*b*)) it is interesting to note that as b varies from zero to $\frac{1}{2}$ L that x varies from $\frac{2}{3}$ L to $\frac{1}{2}$ L. This means that the section of maximum deflection in this case can be displaced only 16.7% of L from the middle of the beam (compare with Case 3).

To determine the points of inflection of this beam write the general shear equation of the mathematical beam and equate to zero; solve the resulting equation for x:

$$\frac{P\,a\,b^2}{L^2} + \left(\frac{P\,a^2 b}{L^2} - \frac{P\,a\,b^2}{L^2}\right)x - \frac{P\,b x}{L} = 0 \quad\text{...........(63)}$$

or,

$$x = \frac{a\,L}{L + 2\,a} \quad\text{.........................(64)}$$

It is to be noted that in Equation (64), x is equal to one-half the distance to the point of maximum deflection (see Equation (62)).

Case 12.—Fixed Beam with Segment of Uniform Load at Any Position.—Applying the method in Case 11, and using the slope of the simply supported beam with a segment of uniform load as in Case 4:

$$\frac{M_1 L}{3} + \frac{M_2 L}{6} = \frac{w}{6\,L}\left[\frac{L^2}{2}(a^2_2 - a^2_1) - \frac{1}{4}(a^4_2 - a^4_1)\right] \quad\text{.......(65)}$$

and,

$$\frac{M_1 L}{6} + \frac{M_2 L}{3} = \frac{w}{6\,L}\left[\frac{L^2}{2}\left\{(L - a_1)^2 - (L - a_2)^2\right\}\right.$$

$$\left. - \frac{1}{4}\left\{(L - a_1)^4 - (L - a_2)^4\right\}\right] \quad\text{................(66)}$$

Solving for M_1 and M_2:

$$M_2 = \frac{w}{3\,L^2}\left[\frac{L^2}{2}\left\{2\,(L - a_1)^2 - 2\,(L - a_2)^2 - a^2_2 + a^2_1\right\}\right.$$

$$\left. - \frac{1}{4}\left\{2\,(L - a_1)^4 - 2\,(L - a_2)^4 - a^4_2 + a^4_1\right\}\right] \ldots\ldots\ldots\ldots (67)$$

$$M_1 = \frac{w}{3\,L^2}\left[\left(\frac{L^2}{2}\left\{2\,(a^2_2 - a^2_1) - (L - a_1)^2 + (L - a_2)^2\right\}\right.\right.$$

$$\left. - \frac{1}{4}\left\{(2\,a^4_2 - 2\,a^4_1) - (L - a_1)^4 + (L - a_2)^4\right\}\right] \ldots\ldots\ldots (68)$$

and,

$$M_1 = \frac{w}{L^2}\left[\frac{L}{3}\,(a^3_2 - a^3_1) - \frac{1}{4}\,(a^4_2 - a^4_1)\right] \ldots\ldots\ldots\ldots (69)$$

Case 13.—Fixed Beam with a Couple Applied at Any Point.—Applying the method developed in Case 11, and using the slope at the end of a simply supported beam loaded with a couple, which was developed in Case 5 (see, also, Fig. 16):

$$\frac{M_1\,L}{3} + \frac{M_2\,L}{6} = \frac{X\,b^3}{6\,L^2} + \frac{X\,a\,b^2}{2\,L^2} - \frac{X\,a^3}{3\,L^2} \ldots\ldots\ldots\ldots (70)$$

and,

$$\frac{M_1\,L}{6} + \frac{M_2\,L}{3} = -\frac{X\,a^3}{6\,L^2} - \frac{X\,b\,a^2}{2\,L^2} + \frac{X\,b^3}{3\,L^2} \ldots\ldots\ldots\ldots (71)$$

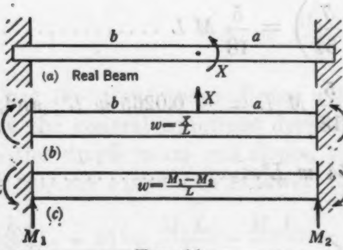

FIG. 16

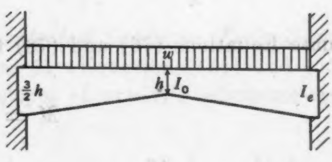

FIG. 17

From Equations (70) and (71),

$$M_2 = \frac{X}{L^3}\,(b^3 - 2\,b\,a^2 - a\,b^2) \ldots\ldots\ldots\ldots\ldots\ldots (72)$$

and,

$$M_1 = \frac{X}{L^3}\,(-\,a^3 + 2\,a\,b^2 + a^2\,b) \ldots\ldots\ldots\ldots\ldots\ldots (73)$$

Case 14.—Fixed Beam with Variable Moment of Inertia and with Uniform Load Over Entire Span.—To solve the fixing moments, the beam in

Fig. 17 is considered simply supported and the end slopes are found. The end moments (unknown) are then applied, bringing the end slopes to zero. The moment of inertia at any section (with the origin at the center) is,

$$I_x = \frac{I_o\,(L + x)^3}{L^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(74)$$

The loading is expressed by the formula:

$$\frac{V}{I} - \frac{M}{I^2}\left(\frac{d\,I}{d\,x}\right) = -\frac{w\,x\,L^3}{I_o\,(L + x)^3} - \frac{3\,w\,L^5}{8\,I_o\,(L + x)^4} + \frac{3\,w\,L^3\,x^2}{2\,I_o\,(L + x)^4} \ldots(75)$$

in which I_o = moment of inertia at the center of the beam (see Fig. 17). The end slope of a simply supported beam is:

$$\phi_L = -\frac{w\,L^3}{I_o}\int_0^{\frac{L}{2}} \frac{x^2\,dx}{(L + x)^3} - \frac{3\,w\,L^5}{8\,I_o}\int_0^{\frac{L}{2}} \frac{x\,dx}{(L + x)^4} + \frac{3\,w\,L^3}{2\,I_o}\int_0^{\frac{L}{2}} \frac{x^3\,dx}{(L + x)^4}$$

$$= \frac{w\,L^3}{I_o}\,(-\,0.0167 - 0.0162 + 0.00645) = -\,0.0265\,\frac{w\,L^3}{I_o} \quad \ldots\ldots(76)$$

For a beam loaded with Moment M at the ends, the intensity of loading equals $\frac{M}{I^2}\left(\frac{dI}{dx}\right)$. Then the end slope is:

$$\phi_L = \frac{M\,L}{I_e^{\,2}} + \int_0^{\frac{L}{2}} \frac{M}{I^2}\,\frac{dI}{dx}\,x\,dx = \frac{4\,M\,L}{27\,I_o} + \int_0^{\frac{L}{2}} \frac{3\,M\,x\,dx\,L^3}{I_o\,(L + x)^4} \quad \ldots\ldots(77)$$

and,

$$I_o\,\phi_L = M\,L\left(\frac{4}{27} + \frac{7}{54}\right) = \frac{5}{18}\,M\,L \quad \ldots\ldots\ldots\ldots\ldots(78)$$

Equate Equations (76) and (78) for ϕ_L: $\frac{5}{18}\,M\,L = -\,0.0265\,w\,L^3$; and,

$$M = -0.0954\,w\,L^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(79)$$

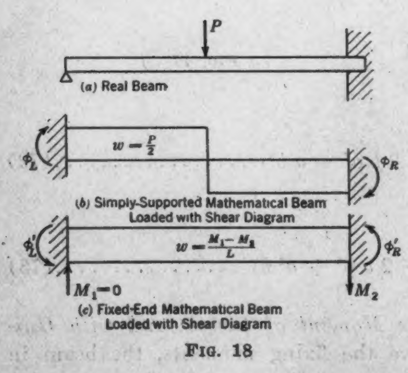

FIG. 18

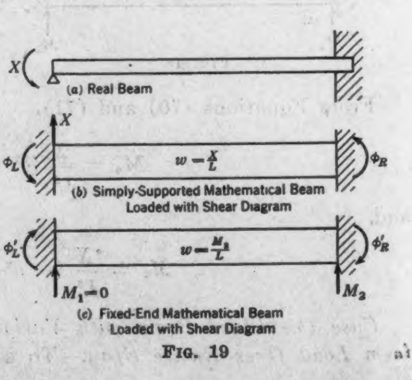

FIG. 19

Case 15.—Propped Cantilever with Concentrated Load at Mid-Span.— Applying the same method as in Case 11 where $M_1 = 0$ (Fig. 18): $\frac{M_2 L}{3} = \frac{P L^2}{16}$; $M_2 = \frac{3}{16} P L$; and,

$$\phi_A = \frac{P L^2}{16} - \frac{M_2 L}{6} = \frac{P L^2}{32 E I} \dots\dots\dots\dots\dots\dots (80)$$

Case 16.—Propped Cantilever with Uniform Load Over Entire Span.— For this case: $\frac{M_2 L}{3} = \frac{w L^3}{24}$; $M_2 = \frac{1}{8} w L^2$; and,

$$\phi_A = \frac{w L^3}{24} - \frac{M_2 L}{6} = \frac{w L^3}{48 E I} \dots\dots\dots\dots\dots\dots (81)$$

Case 17.—Propped Cantilever with Concentrated Load Not at Mid-Span.— When the concentrated load in Fig. 18 is off center: $\frac{M_2 L}{3} = \frac{P a b}{6 L} (L + b)$; and $M_2 = \frac{P a b}{2 L^2} (L + b)$.

Case 18.—Propped Cantilever with Couple Applied at End.—Using Equations (33b) and (55): $\frac{M_2 L}{3} = - \frac{X L^3}{6 L^2}$, when $a = L$ and $b = 0$.

Then,

$$M_2 = - \frac{x}{2} \dots\dots\dots\dots\dots\dots\dots\dots\dots (82)$$

and,

$$\phi_A = \frac{M_2 L}{2} + \frac{X L}{2} = \frac{M_2 L}{4 E I} \dots\dots\dots\dots\dots\dots (83)$$

Case 19.—Continuous Beam with Concentrated Loads in Each Span.— Using the general equations derived in Case 11 (Equations (54) and (55)), with the simple beam end slopes, the end slopes of any two adjoining spans (Fig. 20) are equated as follows:

$$\frac{P_1 a b}{6 L_1} (L_1 + b) - \frac{M_1 L_1}{6} - \frac{M_2 L_1}{3} = - \frac{P_2 c d}{6 L_2} (L_2 + c) + \frac{M_2 L_2}{3} + \frac{M_3 L_2}{6} \;. (84)$$

and,

$$- M_1 L_1 - 2 M_2 (L_1 + L_2) - M_3 L_2 = - \frac{P_1 a b}{L_1} (L_1 + b) - \frac{P_2 c d}{L_2} (L_2 + c) \;..(85)$$

Case 20.—Continuous Beam with Uniform Load Over Each Span.—For the case of an uniformly distributed load over the entire length of a continuous beam:

$$\frac{w_1 L^3_1}{24} - \frac{M_1 L_1}{6} - \frac{M_2 L_1}{3} = - \frac{w_2 L^3_2}{24} + \frac{M_2 L_2}{3} + \frac{M_3 L_2}{6} \dots\dots (86)$$

and,

$$-M_1L_1 - 2M_2(L_1 + L_2) - M_3L_2 = -\frac{w_1L_1^3}{4} - \frac{w_2L_2^3}{4} \quad \ldots\ldots(87)$$

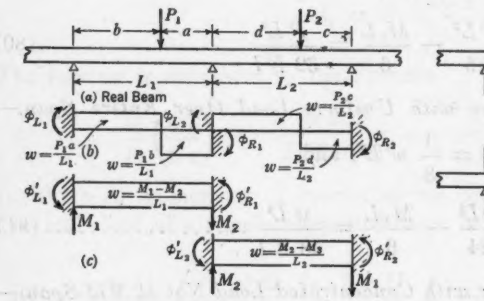

FIG. 20

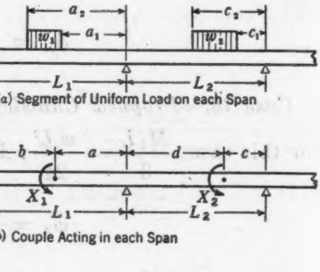

FIG. 21

Case 21.—Continuous Beam with Section of Uniform Load in Each Span (see Fig. (21a)).—Applying the same reasoning as in Case 19, using Equations (30), (54), and (55),

$$\frac{w_1}{6L_1}\left[\frac{L_1^2}{2}\left\{(L_1 - a_1)^2 - (L_1 - a_2)^2\right\} - \frac{1}{4}\left\{(L_1 - a_1)^4 - (L_1 - a_2)^4\right\}\right] - \frac{M_1L_1}{6}$$

$$-\frac{M_2L_1}{3} = -\frac{w_2}{6L_2}\left[\frac{L_2^2}{2}(c_2^2 - c_1^2) - \frac{1}{4}(c_2^4 - c_1^4)\right] + \frac{M_2L_2}{3} + \frac{M_3L_2}{6} \quad ..(88)$$

Therefore:

$$-M_1L_1 - 2M_2(L_1 + L_2) - M_3L_2 = -\frac{w_1}{L_1}\left[\frac{L_1^2}{2}\left\{(L_1 - a_1)^2 - (L_1 - a_2)^2\right\}\right.$$

$$\left.-\frac{1}{4}\left\{(L_1 - a_1)^4 - (L_1 - a_2)^4\right\}\right] - \frac{w_2}{L_2}\left[\frac{L_2^2}{2}(c_2^2 - c_1^2) - \frac{1}{4}(c_2^4 - c_1^4)\right] \quad .(89)$$

Case 22.—Continuous Beam with Couple, X, Applied in Each Span (see Fig. 21(b)).—Applying the same method as in Case 19, using Equations (33a), (33b), (54), and (55),

$$\frac{1}{6}\left(-\frac{X a^3}{L_1^2} - \frac{3 X a^2 b}{L_1^2} + \frac{2 X b^3}{L_1^2}\right) - \frac{M_1L_1}{6} - \frac{M_2L_1}{3}$$

$$= -\frac{1}{6}\left(\frac{X d^3}{L_2^2} + \frac{3 X c d^2}{L_2^2} - \frac{2 X c^3}{L_2^2}\right) + \frac{M_2L_2}{3} + \frac{M_3L_2}{6} \quad \ldots\ldots(90)$$

Therefore,

$$-M_1L_1 - 2M_2(L_1 + L_2) - M_3L_2$$

$$= +\frac{X_1}{L_1^2}(a^3 + 3a^2b - 2b^3) - \frac{X_2}{L_2^2}(d^3 + 3cd^2 - 2c^3) \quad \ldots\ldots(91)$$

CONCLUSIONS

The shear-area method, which uses a type of elastic load, is not suggested as the shortest method for solving slopes and deflections for all problems; but it is suggested as particularly adapted for those problems involving distributed load. In the latter case the static moment and one-half the moment of inertia of the shear area are used more easily than the shear and the static moment of the curved moment area. For the concentrated loads the functions of the shear area are obtained quite as easily as those of the moment area. For a beam with varying moment of inertia the solutions by the shear and moment areas require about the same effort even though the loading for the former, $\frac{V}{I} - \frac{M}{I^2}\left(\frac{dI}{dx}\right)$, appears to be more cumbersome than the loading of the latter, $\frac{M}{I}$.

Any one who is interested in using a simpler elastic load than the shear area, or the shear area modified, can investigate the loading directly. In this case the functions of the slope and the deflection are given by the following equations,

For slope:

$$\phi_x E I = -\frac{1}{2}\int dA\, x^2 + \frac{R x^2}{2} + M_e\, x + \phi_e \;\ldots\ldots\ldots\ldots (92)$$

and,

$$y E I = -\frac{1}{6}\int dA\, x^3 + \frac{R x^3}{6} + \frac{M_e\, x^2}{2} + \phi_e\, x + y_e \ldots\ldots (93)$$

or,

$$\phi_x E I = -\frac{1}{2} I_x + M_e\, x + \phi_e \;\ldots\ldots\ldots\ldots\ldots\ldots (94)$$

and,

$$y E I = -\frac{1}{6} Q_x + \frac{M_e\, x^2}{2} + \phi_x\, x + y_e \;\ldots\ldots\ldots\ldots (95)$$

in which $\int dA = \int w\,dx$, or P; M_e = end moment; ϕ_e = end slope; and y_e = end deflection.

APPENDIX

NOTATION

In the following notation, presented for the convenience of reference, an effort has been made to conform as nearly as practicable with "Symbols for

Mechanics, Structural Engineering, and Testing Materials" advanced by the American Standards Association[8]:

a = distance of a section from the right end of a beam ($=L-b$).
b = distance of a section from the left end of a beam ($=L-a$).
c = distance of a section (corresponding to a) from the right end of Span 2 of a continuous beam ($= L_2 - d$).
d = distance of a section (corresponding to b) from the left end of Span 2 of a continuous beam ($= L_2 - c$).
e = a subscript denoting "at the end."
m = a subscript denoting "maximum."
o = a subscript denoting "origin," or "at the center."
w = load per unit distance; load intensity per foot on a given beam.
x = a distance measured parallel to the X-axis; as a subscript, x refers to Section X–X;
y = deflection at any section, X–X, in a given beam (= slope at any section, X–X, of a mathematical beam); y_m = maximum deflection.
A = area of a loading diagram on a mathematical beam.
C = constants (see Equation (6)).
E = modulus of elasticity.
I = rectangular moment of inertia; I_x = moment of inertia of the shear diagram at any section, X–X; I_o = moment of inertia at the center of a beam; I_e = moment of inertia at the ends of a beam.
L = length; span of a given beam; as a subscript, L denotes "left end."
M = moment of force; bending moment in a given beam; M_T = total moment of force.
P = concentrated load on a given beam.
R = reactions, or resultants; as a subscript, R denotes "right end."
T = a subscript denoting "total."
V = total shear in a given beam.
X = a force couple.
ϕ = slope of the elastic curve; ϕ_L = slope at the left end of a given beam (= bending moment at the left end of a mathematical beam); ϕ_R = slope at the right end of a given beam (= bending moment at the right end of a mathematical beam); ϕ_T = total change in slope; ϕ_x = slope at any section X–X, in a given beam (= bending moment at Section X–X. in a mathematical beam).

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

ANALYSIS OF SHEET-PILE BULKHEADS

Discussion

BY PAUL BAUMANN, M. AM. SOC. C. E.

PAUL BAUMANN,[37] M. AM. SOC. C. E. (by letter).[37a]—The use of high sheet-pile bulkheads is almost entirely limited to water-front engineering and particularly to harbor work. The increase in draft of ocean-going vessels has called for deeper slips and they, in turn, have called for higher bulkheads. This trend toward greater structural dimensions has been present in nearly all fields of engineering for the past two decades and bulkheads of unprecedented height are reasonably certain to be built in the near future. It is this consideration that prompted the writer to present his paper, besides the desire to deliver the theory on passive resistance from the principal obstruction to a deeper insight into the intrinsic behavior of soil, namely, the angle of repose or the angle of internal friction, conveniently but erroneously used as constant values; of substituting simple formulas for the specific, passive resistance in which the classical angle, ϕ, does not appear and the constants of which can only be correctly ascertained by test; and last, but not least, of preventing serious error in analyzing sheet-pile bulkheads without taking into account the relative flexibility of sheet-piling and the relative yield of the ground.

The response which this paper has met is very encouraging indeed. The writer is much indebted to the discussers who, in a truly professional spirit, have contributed so generously to its academic and practical value. Generally, they seem to agree that a step toward a more exact solution of sheet-pile problems is quite timely.

As all the symbols used in the paper were defined when and where they appear, a list of notations was omitted by the writer, due to lack of space, and it is much to his regret that this omission seems to have caused undue difficulties to some of the discussers in studying the paper. The same holds

NOTE.—The paper by Paul Baumann, M. Am. Soc. C. E., was published in March, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: In May, 1934, by Jacob Feld, M. Am. Soc. C. E.; August, 1934, by Messrs. R. L. Vaughn, M. A. Drucker, and Raymond P. Pennoyer; October, 1934, by D. P. Krynine, M. Am. Soc. C. E.; November, 1934, by Dr. Ing. E. h. O. Franzius; December, 1934, by Theodor von Kàrmàn, M. Am. Soc. C. E.; and March, 1935, by Dr. Ing. E. Lohmeyer.

[37] Asst. Engr., in Chg. of Mountain Div., Los Angeles County Flood Control Dist., Los Angeles, Calif.

[37a] Received by the Secretary April 1, 1935.

true of definitions of mathematical deductions which were minimized for the same reason, although as far as the writer is aware, none of the substitutions or transformations, exact or approximate, was made without a brief explanation.

Those who failed to study the paper in all its details inevitably found it difficult to separate the results of: (*a*) The test bulkhead analysis; (*b*) the outer harbor bulkhead analysis; (*c*) the laboratory tests of Professor Franzius; (*d*) the writer's deductions, based on these laboratory tests (including the true distribution of passive resistance, the possible stress distribution in the passive prism, and the differential equation of the surface of rupture); (*e*) the new theory for the design of sheet-pile bulkheads; and, (*f*) the application of this theory to the test bulkhead.

Mr. Feld believes that the end piles in the test wall took less load than the inside piles because of active pressure on the end walls causing an outward movement which, in turn, engaged a passive prism and thereby deprived the earth in front of the end piles of some of its resistance. This is contrary to the opinions of Professor Franzius and Professor Krynine who conclude that the end piles had more support in the ground than the other piles because of the wedges that "fan out" laterally beyond a vertical plane through the side walls. In principle, the writer agrees with Professor Franzius and Professor Krynine so far as an addition to, rather than a deduction from, the support of the end piles is concerned. However, Mr. Feld is correct in concluding from Fig. 6' that the end piles took less load than the inside piles; but this was due to fast percolation between the test wall and the side walls, owing to the short penetration of the copper seals. This short penetration actually caused a blow-out shortly after the full head was reached and was kept constant until the deflection was measured, which put the test results, both as to passive resistance and interlock efficiency, on the side of safety.

Naturally, such discrepancies are not desirable, but under the circumstances it was quite impossible to prevent them, because, this was a field test, hurriedly prepared in connection with a lawsuit and not a carefully prepared laboratory experiment. It is hoped that this statement will answer objections made by Messrs. Feld, Vaughn, Pennoyer, Franzius, and Krynine in regard to shortcomings of the test bulkhead.

In his paper[38] on "Lateral Earth Pressures," Mr. Feld states: "It should be noted that ϕ' [angle of friction between earth and wall] decreases as the height of the fill increases * * *." Furthermore, he consistently found the center of pressure to be above the third-point of the height of the fill, indicating non-linear (parabolic) distribution of pressure and, therefore, a variable angle of internal friction. Nevertheless, he cannot agree with any of the writer's conclusions.

In discussing the use of the angle of repose instead of the angle of internal friction, Messrs. Feld and Vaughn seem to overlook the fact that the angle of repose of the material under water was used. This angle of repose is substantially the same as the angle of internal friction under water

[38] *Transactions*, Am. Soc. C. E., Vol. LXXXVI (1923), p. 1470.

as capillary action or cohesion is eliminated thereby. The determination of the angle of internal friction of the material in the dry, as suggested by Mr. Feld, is probably subject at least to as much error as the determination of the angle of internal friction of the material under water, as arrived at by the writer.

In stating that the writer's new theory "only applies within very small limits, practically within elastic deformations of the soil structure," Mr. Feld evidently was misled by the writer's use of the wording, "elasticity of both wall and soil," which was used in its descriptive rather than in its physical meaning. The theory applies fully within the limits of the tests through which the constant, b, m, and n, are determined, whereas the practical limits naturally are governed by a safety factor. Thus, if the tests should show that a wall movement of y in. is necessary to cause failure of the soil structure along the surface of rupture, only a fraction of y would be allowed in practice. As the deflection diagrams (that is, the elastic lines) can be drawn to such scales as will greatly magnify the distance, y, the smallness of these movements is of no concern.

The writer agrees with Mr. Feld that there is no method for measuring passive resistance for zero movement, because there is none, but he does not agree as far as active pressure is concerned, provided zero is considered from a standpoint of testing (that is, relative pressure as against absolute pressure).

Mr. Vaughn gives an excellent treatise on dredging. However, his claim that relieving platforms are not effective is not supported by accounts of European harbor developments where this type was originated and has since been used extensively. It is true though that to make relieving platforms effective, considerable care is required, particularly during the filling through the holes. The grain diameter of the fill material behind the bulkhead at Berth 3 at Long Beach was generally less than 0.1 mm and, therefore, was of such size as to lend itself to a "quick" condition and in this state substantially acted as a liquid. Relieving platforms are predicated on stable fills of reasonable density and permeability which is physically impossible to accomplish within the period of construction with material of such fineness.

Mr. Vaughn believes that there is cohesion in soil under water, but at the same time he admits that when dredged vertically the bank does not stand up any length of time, but collapses just like granular soil. The answer is that there is no appreciable cohesion in soil under water. He compares resistance factors in passive prisms with sliding factors in hydraulic-fill dams and seems to be under the impression that they should be the same. The former is expressed by the term, $\tan\left(45 + \dfrac{\phi}{2}\right)$, and the latter by $\tan \phi$, which cannot be equal. Probably, he was thinking of active pressure, in which case $\tan \phi = \tan\left(45 - \dfrac{\phi}{2}\right)$, for $\phi = 30$ degrees. He has overlooked the fact that the passive resistance of a prism is always greater than its weight as long as the bottom of the wall is below the lowest ground surface. For level

ground and a frictionless wall, the ratio, $\frac{E_p}{G} = \tan\left(45 + \frac{\phi}{2}\right) = 1.73$ for $\phi = 30°$, whereas the analysis shown in Fig. 7 gave $\frac{E_p}{G} = \frac{20\,400}{14\,650} = 1.40$ for a sloping ground and a rough wall, which is consistent. Thus, before casting doubt on the validity of this latter factor Mr. Vaughn should disprove the validity of the classical theory of passive resistance of earth.

Mr. Drucker refers to the use of the factor, 2, which was applied to the Coulomb formula and describes this procedure as an assumption that the actual passive resistance is twice the theoretical. This is incorrect. The fact is that the actual passive resistance for a rough wall was assumed to be twice the theoretical for a frictionless wall, according to Coulomb's formula. The analysis of the test bulkhead shows an effectiveness factor of 1.68 in spite of the weakening of the passive prism through percolation at the ends, as mentioned before, and a reduction in wall friction in addition.

Mr. Drucker refers to Equation (69) which is from a paper[19] by R. P. Pennoyer, Assoc. M. Am. Soc. C. E., and shows correctly, by mathematical deductions, that the points of zero moment and zero load do not necessarily coincide. The writer fully agrees with Mr. Drucker and always did agree as may be seen from Equations (2) and (3) and the intervening text. Throughout all the analyses the distance, a_0, between the ground surface and the point of zero load was kept apart from the distance, X_0, between the ground surface and the point of zero moment. The term, 0.06 h_0, in Equation (3) is, in fact, nothing but the approximate equivalent of 0.60 X_0.

Thus, Equation (69) and all of Mr. Pennoyer's deductions, referred to by Mr. Drucker, are not general, but are based on the special case, $a_0 = X_0$.

Mr. Drucker's Equation (83) is a fair approximation for d, the exact equation, as derived by the writer, being:

$$d = \frac{0.235\, t^5_1 - 0.532\, t^4_1\,(a_0 - X_0) - 0.69\, t^3_1\,(a_0 - X_0)^2}{0.59\, t^4_1 - 1.59\, t^2_1\,(a_0 - X_0)^2 + (a_0 - X_0)^4}$$
$$+ \frac{0.437\, t^2_1\,(a_0 - X_0)^3 + \frac{t_1}{2}\,(a_0 - X_0)^4 + \frac{1}{3}\,(a_0 - X_0)^5}{0.59\, t^4_1 - 1.59\, t^2_1\,(a_0 - X_0)^2 + (a_0 - X_0)^4} \quad \ldots\ldots(130)$$

Mr. Drucker doubts the scientific background of the writer's new method of analysis because he believes it to be "based primarily on an arbitrarily assumed point of zero moment below the ground surface which, in turn, determines the passive resistance distribution for the entire depth of embedment." Indeed, if such were the case, the method would be not only unscientific, but worthless. However, such is not the case, and the point of zero moment is, in fact, the final answer in the problem. In the paragraphs following Equation (67) the writer made evident, that the line of closure of the moment diagram is assumed, subject to correction, and that it may serve as an aid to the designer to know that the true point of zero moment is gen-

[19] "Design of Steel Sheet-Piling Bulkheads," *Civil Engineering*, November, 1933, p. 615.

erally closer to the surface than 0.10 h_0 due to the parabolic distribution of the passive resistance, at least as far as steel sheet-piling is concerned.

It is hoped that the analysis of a high bulkhead, presented subsequently herein will serve to clear up most of Mr. Drucker's questions and to disprove Dr. Lohmeyer's contention that the new method of analysis is substantially the same as that of Dr. Blum. The two initial assumptions in the new method (namely, the line of closure in the moment diagram and the line of reference in the deflection diagram) are subject to correction until the one and only position of each line is found which will satisfy the equilibrium conditions. For a given active loading and specific passive resistance, $e_p = f\ (z, y)$, there is one and only one correct position for these lines, and the corresponding load and moment distributions follow automatically and are not subject to arbitration. In spite of painstaking efforts, Mr. Drucker did not succeed in approximating the load distribution as obtained by the new theory. This would be particularly evident if his pressure lines in Fig. 18 had been drawn to a point 6 in. below the ground surface as they should have been.

Contrary to the writer, Mr. Drucker computes the factor of safety against "push-out" (Fig. 7) on the basis of total, rather than differential, loading. In other words, he assumes that the criterion of this safety is governed by the ratio of ultimate and existing, active loading. This may prove advantageous in certain cases. However, if the active loading is definite (as it is generally), it would seem to the writer that the more logical answer to the question of safety against "push-out," is to determine how many times the existing, passive resistance could be reduced before failure would occur. This is the way the writer arrived at his factor of 2.42.

Both Mr. Vaughn and Mr. Pennoyer take exception to the writer's statement that 75% interlock efficiency was advocated by American manufacturers. Regretfully, the writer admits that he has never had the pleasure of discussing this question with American manufacturers. However, he heard this statement during interviews with sales representatives of American manufacturers, and he wishes to apologize for having taken it for granted.

In answer to Mr. Pennoyer's query, the writer wishes to emphasize that Equation (3) is quoted from "Der Grundbau."[3] This should explain the subsequent statement in the paper regarding its value. The same applies to Mr. Pennoyer's Equation (95), as was outlined before, namely, that it applies to one special case only; that is, $a_0 = X_0$. Contrary to Mr. Pennoyer's claim, the writer does not advocate that, invariably, the designer should assume a 75% interlock efficiency; he actually points out what requisites are necessary to justify such an assumption.

The writer agrees with Dr. Lohmeyer that failures of steel sheet-pile bulkheads designed on a basis of 100% interlock efficiency, as has been customary in many places in the world, were due to "push-out" rather than to failure of the piles. As to his doubts concerning the justification of a new theory, Mr. Pennoyer is referred to the comparative analyses introduced subsequently herein.

[3] "Der Grundbau," von Brennecke-Lohmeyer, Fourth Edition, 1930, Vol. II, p. 77 *et seq.*

Mr. Pennoyer and Dr. Lohmeyer are correct in their claim that the determination of the interlock efficiency and of the maximum stresses is approximate only and of qualitative rather than of quantitative value.

Dr. Lohmeyer's exact analysis of stresses and deflections under the influence of slippage in the interlock is a real contribution to research in this field. Whether or not the solution is as complete from a physical standpoint as from a mathematical standpoint remains to be shown by tests. It is quite possible that the occurrence of opposite stresses at points in contact in the interlocks has an influence which is not reflected in the analysis. As may be seen from Fig. 10 the writer was aware of the fact that the stress limits are due to zero friction (single sheet) and zero slippage (interlocked sheets), but he was not aware of the great deviation from a linear relation between the two, as brought out by Dr. Lohmeyer.

Several years ago the writer approached the analysis of the effect of clearance in the interlock on the basis of compound beam action (differential curvature), but found the influence on the stresses quite small. This differential curvature causes a small restraining effect at or near the supports which tends to reduce the field moment. For a compound beam on two supports, acted upon by a uniform load, the formula of the moment due to differential curvature, to be superimposed on the moment for a simple beam, is,

$$\Delta M = \frac{2.655\, I\, e'\, f^2}{e''\, E}\left(\frac{20\, X^4}{l^4} - \frac{40\, X^3}{l^3} + \frac{18\, X^2}{l^2} + \frac{2\, X}{l} - 1\right)$$

in which l = the span length; e' = distance, in inches, to outside web fiber; e'' = distance, in inches, to center of interlock from axis of gravity of single sheet; and, f = allowable fiber stress, in pounds per square inch. If interlocked sheet-piling is used as a horizontal beam on two supports, and the loading is resting on the upper sheets only, a condition of practically zero interlock efficiency may be produced which, although of theoretical interest, is in no way related to the actual conditions of bulkheads.

The most important feature brought out by Dr. Lohmeyer is the relatively small increment in fiber stress with a relatively great decrement in shear transmission in the interlock. Thus, in using the ratio between the actual outside fiber stress and the theoretical outside fiber stress for zero slippage as interlock efficiency the latter would show higher values, based on Dr. Lohmeyer's exact method, than those based on the writer's approximate method.

With the loading, moment distribution, and deflections known, the interlock efficiency can be determined directly by Dr. Lohmeyer's method, provided the sheets are not stressed beyond the elastic limit.

Professor Krynine's discussion displays an intimate knowledge of soil mechanics and deals, in a fundamental and constructive manner, with the essentials only. There is no doubt but that Equation (96) furnishes a better and more logical approximation of the elastic line than the hyperbolic spiral used by the writer and, due to its simplicity, would be well suited for introduction in the analytical solution as outlined in Equations (61) to (66), inclusive.

The zone of greater density next to a pile driven into a granular mass, so far as its dimension normal to the axis of the pile is concerned, is governed entirely by the thickness of the pile and the density of the material. If a granular mass of high density is produced by suitable rolling or tamping at the proper water content or, if it is due to consolidation under load in such a manner as to cause its particles to assume the closest position possible, its behavior will be quite different from a plastic mass such as clay of the same density. A pile driven into such a granular mass would engage an envelope which would be many times as great as that for the same pile driven into clay. This is clearly demonstrated in a simple way by checking the penetration of a plasticity needle, as outlined by R. R. Proctor, Assoc. M. Am. Soc. C. E.,[39] in granular soils and clays. It may be found that with granular material the size of the cylindrical container has a considerable influence even if its diameter is many times the diameter of the needle, whereas with clay the particles seem to be able to slip by each other due to their flaky shape without any appreciable influence on the particles a few needle diameters away. This classification of soil by means of a plasticity needle appears very promising indeed, and it is quite possible that before long the needle will obviate elaborate and costly tests for the determination of soil constants.

If the outside piles had had more support in the ground due to the sidewise extension of the passive prism (which would have been the case had excessive percolation been prevented through the extension of the copper seals to the bottom of the sheet-piling), the inner piles would have thrown some of their load on these outer piles and the result would have been a more uniform deflection and a more uniform, permanent set. However, evidence of percolation appeared under a head of 12 ft—that is, for a gradient of a little more than unity—and it increased rapidly as the maximum head of 15 ft was reached. This feature had been foreseen and was considered desirable for the purpose of securing results on the side of safety.

As a check on the conclusion regarding the decrease of the angle of internal friction with the depth, the writer determined the constants, m, n, and b, in Equation (27) from Professor Franzius' experiments on a frictionless wall as follows: $m = 1.20$; $n = 3.20$; and $b = 1.37 \times 10^{-6}$, which are the values used in the following example of bulkhead analysis.

The angle of internal friction, as a function of z for $y = 0.30$ ft, is shown in Fig. 27 which was plotted from Equation (33).

The equation of the surface of rupture for $y = 0.30$ ft was derived from the differential equation $\frac{d\eta}{dz} = \sqrt{\xi}$, namely,

$$\eta = 3.47\,((H - h)^{0.80} - z - h)^{0.80}) + 0.0325\,((H - h)^{1.20} - (z - h)^{1.20}) \quad \text{..(131)}$$

which is shown graphically in Fig. 28. It is interesting to note that the theoretical surface of rupture is nearly straight in spite of a variable angle, ϕ. It is slightly convex upward—that is, opposite the surface of rupture for a rough wall. The writer wonders if the straight surfaces generally found by

[39] "Fundamental Principles of Soil Compaction," *Engineering News-Record*, August 31, September 7, 21, and 28, 1933.

tests on frictionless walls may not be due to the fact that the walls are never quite frictionless and that the slight friction which cannot be eliminated, is just enough to overcome the upward convexity.

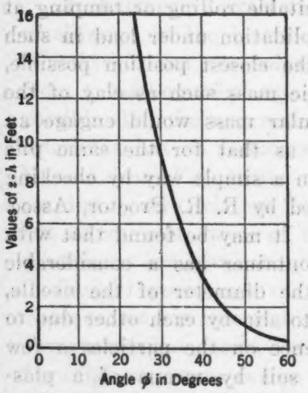

FIG. 27.—ANGLE OF INTERNAL FRICTION, ϕ, FOR $y = 0.30$ FEET.

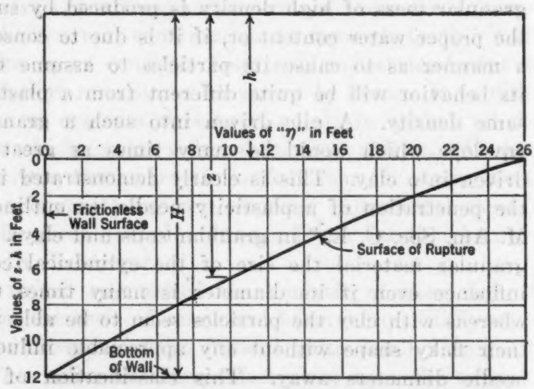

FIG. 28.—SURFACE OF RUPTURE FOR FRICTIONLESS WALL.

In an interesting and convincing manner, Professor von Kármán points out that $\frac{1}{3}\, y_0\, p^3$ should be introduced in the second term of Equation (63) instead of $\frac{1}{2}\, y_0\, p^3$. The writer is appreciative of this correction.

Illustrative Example.—The question raised by several discussers as to whether the writer's new theory is justified, is best answered by means of an example. In Figs. 29, 30, and 31, the analyses of a bulkhead are shown, on the assumption that either steel or reinforced concrete sheet-piling is to be used, having the properties as shown in Table 7.

TABLE 7.—PROPERTIES OF SHEET-PILING; ILLUSTRATIVE EXAMPLE

Material	Moment of inertia, I, in inches ^{4}	Section modulus, S, in inches ^{3}	Modulus of elasticity, E, in pounds per square inch	Product, EI, in inch^2-pound units
Steel...........	673.28	77.50	29×10^6	$19\ 560 \times 10^6$
Concrete...........	27 000	1 800	3×10^6	$81\ 000 \times 10^6$

For purposes of comparison the bulkhead is analyzed on the basis of the Blum-Lohmeyer as well as the writer's method and the following soil properties are used: For the Blum-Lohmeyer: $\phi = 30°$; $\gamma = 100$ lb; and, the effectiveness factor is 2; and for the writer's method: (constants computed from Franzius' test for a frictionless wall):

$$e_p = \frac{160 \sqrt{y}}{\sqrt{1.37 \times 10^{-5}}} (z - h)^{0.60} \quad \text{.........................(132)}$$

and the effectiveness factor is 2.5. The factor, 2.5, was used to make the two methods of analyses directly comparable. Values of e_p in terms of the path, y, and the depth, $z - h$, are shown in Fig. 32.

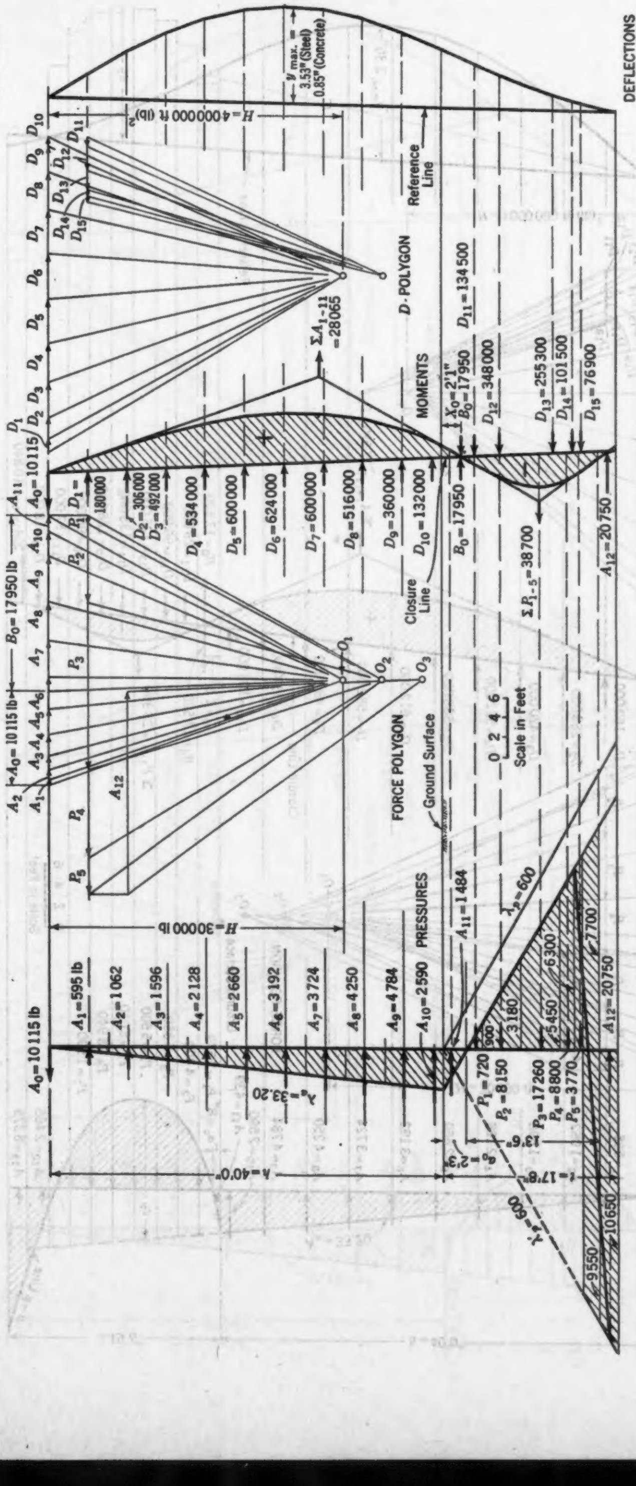

FIG. 29.—ANALYSIS OF A BULKHEAD BY BLUM-LOHMEYER METHOD.

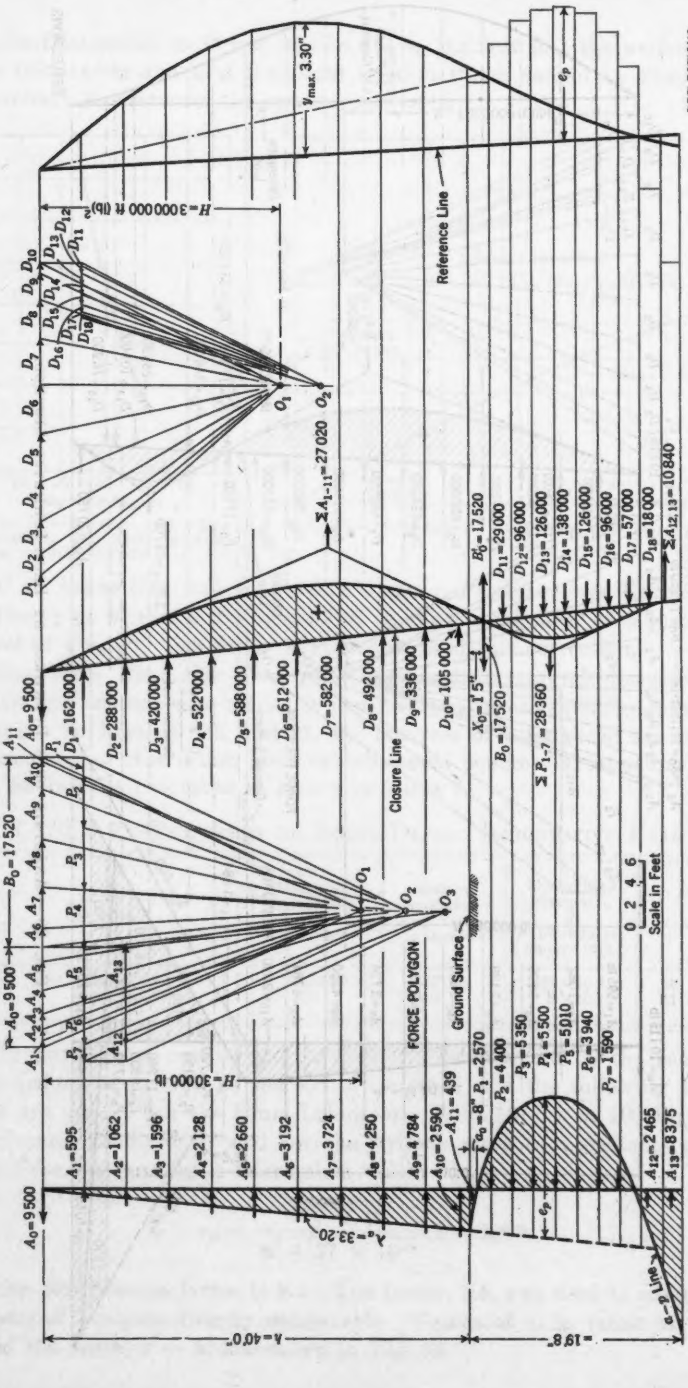

FIG. 30.—ANALYSIS OF STEEL SHEET-PILE BULKHEAD BY BAUMANN METHOD.

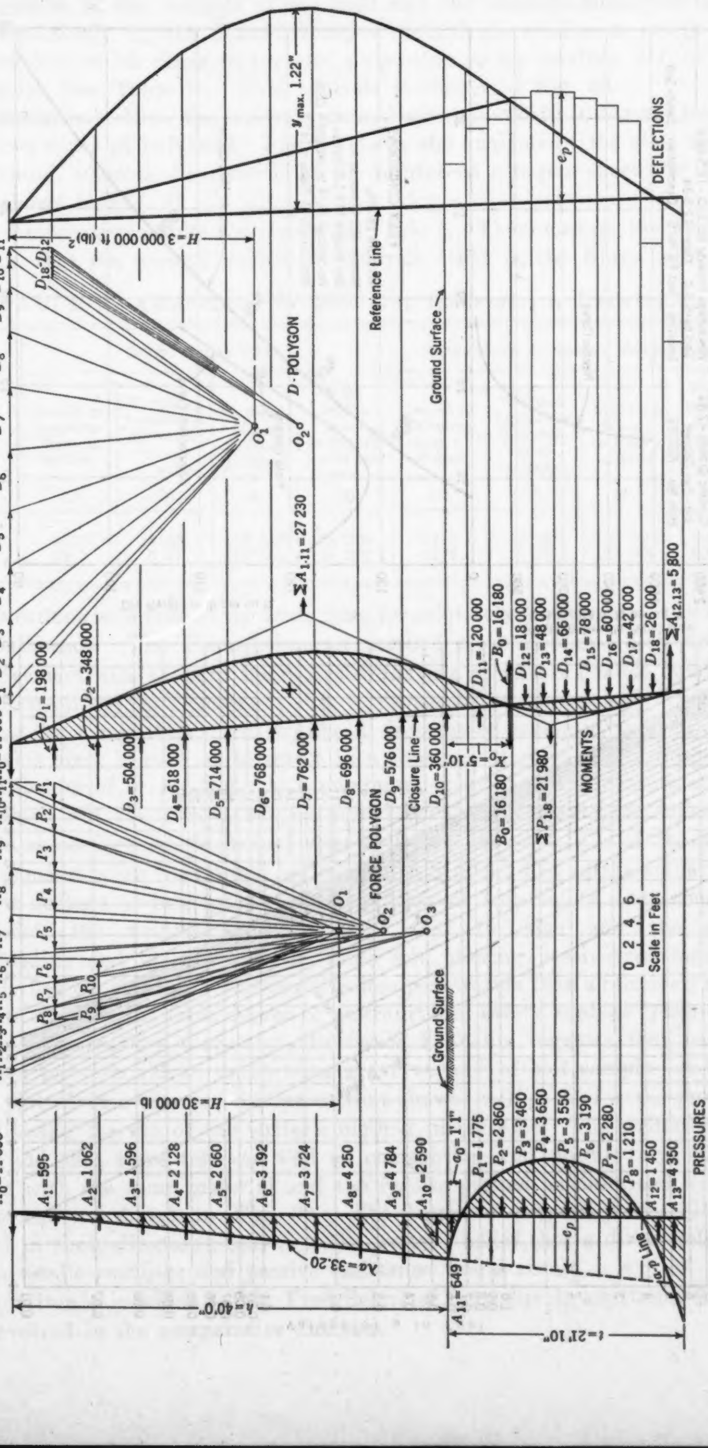

FIG. 31.—ANALYSIS OF CONCRETE SHEET-PILE BULKHEAD BY BAUMANN METHOD.

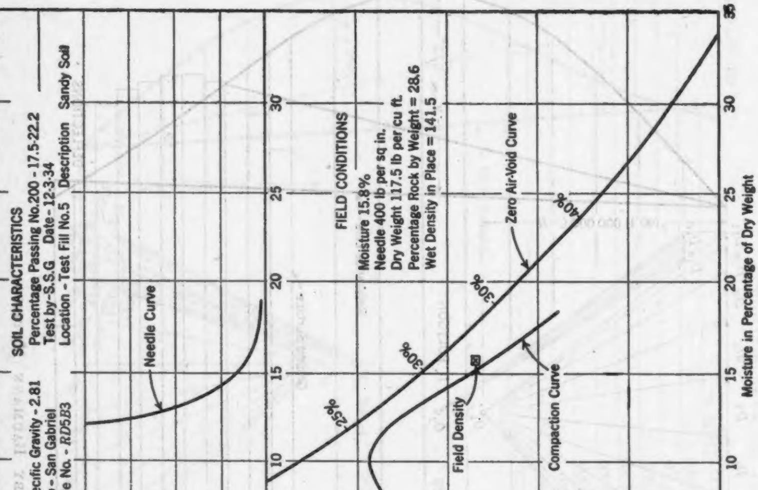

FIG. 33.—EXAMPLE OF SOIL TEST DIAGRAM

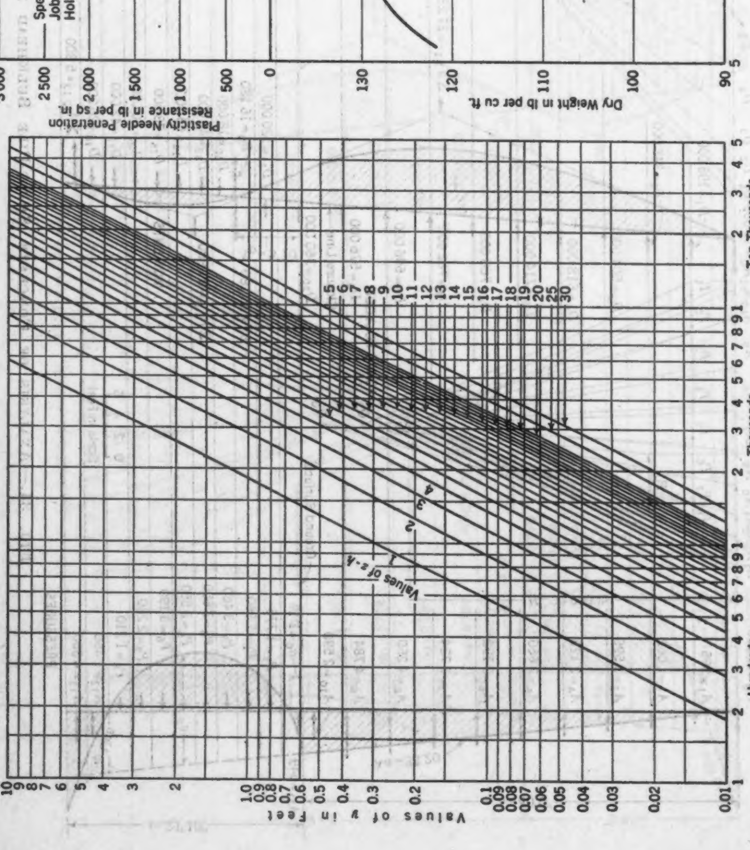

FIG. 32.—RELATION BETWEEN e_p AND y FOR CORRESPONDING VALUES ($z - h$) BASED ON FRICTIONLESS WALL.

The results of the analysis of the steel and the concrete sheet-pile bulkheads, respectively, by the Blum-Lohmeyer method are analogous except for the deflections, which differ in inverted proportion to the product, $E I$, of the two sections (see Table 7). This analysis is shown in Fig. 29.

In contradistinction, the writer's method gives markedly different results for the two types of bulkhead. Fig. 30 shows the analysis of the steel sheet-pile bulkhead, whereas the analysis of the reinforced concrete sheet-pile bulkhead is shown in Fig. 31.

The comparative results are shown in Table 8. They confirm the writer's prediction that the present method of analysis (that is, the Blum-Lohmeyer method) cannot hold true at the same time for relatively flexible, and relatively rigid, bulkheads. The Blum-Lohmeyer method gives results which are in close agreement with those of the writer so far as a relatively flexible bulkhead (steel sheet-piling) is concerned; but it proves to be dangerously in error if applied to a relatively rigid bulkhead (reinforced concrete sheet-piling). Indeed this error is such as to result in a shortage of penetration, necessary for equilibrium of nearly 20%; in a maximum moment and a maximum fiber stress which are about 80% of the actual ones; in a maximum deflection which is about 70% of the actual one; and in a distance, X_0, of the point of zero moment below the ground surface which is about 40% of the actual one. This is a feature of interest from a standpoint of both safety and economy.

TABLE 8.—COMPARISON OF RESULTS; ILLUSTRATIVE EXAMPLE

Method of analysis	Steel Sheet-Piles				Reinforced Concrete Sheet-Piles			
	Depth of penetration, t, in feet	Variable, horizontal distance, y_{max}, in inches	Moment, M_{max}, in foot-pounds	Fiber stress, f_{max}, in pounds per square inch	Depth of penetration, t, in feet	Variable, horizontal distance, y_{max}, in inches	Moment, M_{max}, in foot-pounds	Fiber stress, f_{max}, in pounds per square inch
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Blum-Lohmeyer....	17.7	3.50	159 300	24 700	17.7	0.85	159 300	885
Writer......	19.7	3.37	156 000	24 200	21.8	1.22	192 000	1 070

In brief, the writer's analyses show that for one and the same active loading and for one and the same soil, offering passive resistance, a relatively flexible sheet-pile wall requires less penetration than a relatively rigid sheet-pile wall for the same degree of restraint and safety against "push-out."

In his constructive discussion, Professor Franzius suggests that passive resistance tests should be made on a great variety of soil samples and the results plotted in a manner similar to that shown in Fig. 15 for the purpose of facilitating the use of the writer's method in practice. The writer would like to add that sieve tests as well as compaction and needle tests should be made with the same material and the results plotted as described by Mr. Proctor and as shown in Fig. 33. The intrinsic properties of soil are reflected in these diagrams, and it is the writer's belief that a direct relation between needle readings and passive resistance would result.

The writer is indebted to Mr. Fred Schulhof for valuable assistance in the work involved in the comparative analyses.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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DISCUSSIONS

SAND MIXTURES AND SAND MOVEMENT IN FLUVIAL MODELS

Discussion

By Hans Kramer, M. Am. Soc. C. E.

Hans Kramer,[56] M. Am. Soc. C. E. (by letter).[56a]—The discussions submitted have proved most gratifying, in that they constitute a fairly complete cross-section of science, representing, as they do, the analytical, the experimental, and the applied branches. A number of the points raised by the discussers were covered in the original German manuscript and in the unabridged translation,[57] but unfortunately could not be included in the published paper because of space limitations.

The validity of the du Boys expression for tractive force was the subject of comment in all the discussions, except that by Mr. Keays, with Messrs. Leighly, Matthes, Tiffany and Bentzel, and Straub favoring its use, and Messrs. Thompson, Tchikoff, Mavis, and O'Brien and Rindlaub questioning its value.

Mr. Tchikoff is the most emphatic in his rejection of the du Boys concept, and lists four limitations which he believes to be so serious as to annul the entire theory. In answer to these four limitations, the following pertinent facts may be stated:

1.—In most streams the assumption of infinite width is reasonable. In the Mississippi River, for instance, with average widths of 3 000 to 6 000 ft and average depths of 10 to 50 ft, the width-depth ratio, for all practical purposes, can be considered infinite.

2.—Successful applications of the du Boys expression have been made with materials much larger than 10 mm in grain sizes. Perhaps the most striking example is the experiment performed at the U. S. Waterways Experiment Station with regular tetrahedral blocks 10 in. and 5 in. on a side, in which

Note.—The paper by Hans Kramer, Assoc. M. Am. Soc. C. E., was published in April, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by Messrs. John Leighly, Paul W. Thompson, and Gerard H. Matthes; September, 1934, by Messrs. R. H. Keays, and F. T. Mavis; November, 1934, by Messrs. V. V. Tchikoff, Morrough P. O'Brien and Bruce D. Rindlaub, and Herbert D. Vogel; December, 1934, by Joseph B. Tiffany, Jr., Jun. Am. Soc. C. E., and Carl E. Bentzel, Esq.; and January, 1935, by Lorenz G. Straub, Assoc. M. Am. Soc. C. E.

[56] Capt., Corps of Engrs., U. S. Army; Asst. to Dist. Engr., Memphis, Tenn.

[56a] Received by the Secretary April 15, 1935.

[57] On file at Engineering Societies Library, 29 West 39th Street, New York, N. Y.

an excellent verification of the du Boys expression was obtained.[58] Mr. Matthes in his discussion tells of successful experiments with gravel up to 1 in. in longest dimension, in which the du Boys theory was used.

3.—Most alluvial rivers have the "usual moderate slopes" necessary for the validity of the law of constant critical tractive force. The average slope of the Mississippi River in its lower reaches is less than 0.5 ft per mile, or 0.000095, which is several times less than the slopes used in the experiments.

4.—The fact that the type of flow—whether laminar, turbulent, shooting, or streaming—does not affect the theory is hardly a limitation. Rather, it demonstrates that the du Boys theory is sufficiently inclusive to cover all conditions of flow.

It is realized that there are some discrepancies between the theoretical assumptions of the du Boys expression and actual occurrences. As Professor Straub has pointed out, however, since the equation gives satisfactory results when the proper empirical coefficients are introduced, its use is justifiable even though the motion occurrences are somewhat different from those assumed in the theory.

Professor Leighly has analyzed the significance of the M-value, or uniformity modulus, as a measure of the denseness of a sand mixture. He believes, however, that M is given too much weight in the ultimate equation (Equation (10)), which expresses critical tractive force in terms of the characteristics of the gradation curve, and suggests that "the derivation of these terms must begin from the variations seen in the results of experiment when mean diameter is held constant or is eliminated by computation * * *." In this connection, it was pointed out in both the German original and in the unabridged translation that the scarcity of comparable data at the time these experiments were made precluded the possibility of this procedure, and made it necessary to make an assumption on the basis of existing data as to the weight which should be given to M. The plotting of the data led to the tentative conclusion that the first power of M provided a reasonable weight for this factor. Further studies of critical tractive force now (1935) being completed at various hydraulic laboratories should provide information from which a refinement of Equation (10), including a more nearly exact evaluation of the effect of voids ratio, can be made. At the present writing conclusive data are not available.

It is to be noted that Professor Leighly's analyses of several curves in Fig. 13 fail to take into account the classification of the comparability of the data given in Column (2) of Table 3. In particular, the data relating to Mixtures J and L are given weights, 3 and 4, respectively, in the classification; Mixtures B to H and S to V, likewise, are classified as of secondary or less importance, so far as their comparability with the basic experiments is concerned.

Lieut. Thompson believes that the rate of bed-load movement, rather than the critical tractive force, is the important factor to be considered in

[58] "Investigations of Certain Proposed Methods of Bank and Embankment Protection," *Paper 12*, U. S. Waterways Experiment Station, July, 1933.

the regulation of alluvial streams. In contrast with his opinion is that expressed by Mr. Matthes, who emphasizes the importance of knowing the conditions under which movement commences and ceases. Lieut. Thompson also states that the relations between the critical tractive forces of given sands are not the same as those between their rates of movement, basing his statement on data secured by Lieut. K. D. Nichols at the U. S. Waterways Experiment Station.[58] The experiments cited by Lieut. Nichols are of questionable value because they were made with fairly fine sand mixtures, at relatively low tractive forces, where the influence of riffle formations tended to retard the movement of the finer mixtures. Later information secured at the Experiment Station[59] indicates (1) that at higher tractive forces, after the riffles have been smoothed out, the relations between the rates of movement are similar to those between the critical tractive forces; and (2) that the rate of movement can be very definitely related to the du Boys expression for tractive force.

Mr. Matthes' remarks are especially noteworthy, since they reflect the views of a practical engineer of wide experience in the field of river regulation. It must be kept in mind constantly that even though experimenters and scientists have not yet been able to discover all the laws governing "geschiebe" movement, practical engineers engaged in controlling navigable streams must make use of all the available information that is susceptible of direct application. Mr. Matthes shows that the du Boys expression provides a workable basis for the application of existent data to the problem of river regulation.

The need for a closer correlation of hydraulic model results with the corresponding natural phenomena is emphasized by Mr. Keays. The paper by Herbert D. Vogel, M. Am. Soc. C. E., entitled, "Hydraulic Laboratory Results and Their Verification in Nature,"[60] contains several definite comparisons of this type. As an extension of the information presented by Lieut. Vogel, the U. S. Waterways Experiment Station is actively collecting verification data from all the principal hydraulic laboratories in the United States and in Europe. These data, when assembled and published, should be of great value both to practical hydraulic engineers and to experimenters in the laboratories, in judging the reliability of model studies.

The question of Cartesian *versus* semi-logarithmic or logarithmic plotting of gradation curves is brought up by Professor Mavis. The semi-logarithmic method is undoubtedly preferable for clay and silt mixtures. In these studies, however, the Cartesian method had very definite advantages, where a planimeter was used to measure the areas involved in the calculations of A, A_A, A_B, and M. The use of semi-logarithmic paper would have presented an unnecessary complication. Further discussion of the relative merits of the two methods is contained in the unabridged translation of the original German manuscript.

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[58] *Proceedings*, Am. Soc. C. E., February, 1934, p. 280, Table 4 and Fig. 6.

[59] "Studies of River Bed Materials and Their Movement, with Special Reference to the Lower Mississippi River," *Paper 17*, U. S. Waterways Experiment Station (Publication pending).

[60] *Proceedings*, Am. Soc. C. E., January, 1935, pp. 57–73.

The three assumptions which Professor Mavis states were tacitly made are believed to be substantially true. Inasmuch as the experiments were performed with uniform flow in a straight flume, cross-currents were negligible, and the velocity distributions were virtually identical at all cross-sections. Furthermore, since the sand mixtures were from the same source, the shape and specific gravity of the particles were the same for all three mixtures. It is believed to be a fair assumption that the form-factor may be disregarded in considering ordinary quartz sands.

Curves for Chezy's C, similar to those presented by Professor Mavis for Manning's n in Fig. 16, were plotted in connection with the initial studies, but were not published. The computed C-values were presented in Table 1, Column (11). Professor Mavis' conversion of these values affords a valuable extension of the basic data.

In addition to his discussion of the validity of the du Boys theory, Mr. Tchikoff attacks the problem of "geschiebe" movement from the standpoint of the kinetic flow factor. He does not make clear his definition of silt, however, nor does he explain why he believes it necessary that the "silt" in the model should be the same as the material in Nature. As Lieut. Vogel has pointed out,[61] a model should usually be designed to fulfill but one mission. If bed-load movement is the controlling factor, the bed-load material in the model should be selected to give similarity of action. This similarity is dependent on the selection of the bed material and on the control of the depths and slopes in the model.

Mr. Tchikoff apparently assumes that a stream which is not scouring is inactive as far as bed-load movement is concerned. It is well known, however, that even in channel beds that maintain the same configuration from year to year there is considerable bed-load movement. The material carried out of any given reach is continuously being replaced by material from up stream, with a consequent stability of bed configuration.

It is believed that Mr. Tchikoff has presented an erroneous analysis of the process of bank scouring. He states that the upper parts of the bank are washed out and settle to the bottom. Actually, the upper bank caving is secondary, and usually results from the undermining of the under-burden by the higher tractive forces at the bottom.

An evaluation of tractive force based on the distribution of velocities through the cross-section is suggested by Professor O'Brien and Lieut. Rindlaub. This method is probably sound theoretically, but it entails practical difficulties which, thus far, have precluded its use. The accurate measurement of velocities near a movable stream bed or model bed is almost impossible. Any instrument that is placed close enough to the bed to measure the bottom velocities will induce local scouring, which, in turn, causes eddies and rollers which nullify the reliability of the measurement.

It should be noted, in connection with the statement by Professor O'Brien and Lieut. Rindlaub that side-wall friction has an appreciable resistance to flow, that the walls of the flume used in the experiments were roughened with

[61] *Proceedings*, Am. Soc. C. E., October, 1934, p. 1226.

grains of sand, sprinkled on a coating of wet oil paint to which the grains adhered after the paint dried. The resulting wall roughness was about equal to that of the bed.

Lieut. Vogel's discussion, although general in nature, is significant in view of his extensive experience as Director of the U. S. Waterways Experiment Station. His remarks serve to complement the other discussions, submitted by scientists and by practical river engineers.

Most of the criticisms voiced by Messrs. Tiffany and Bentzel arise from the fact that their experiments were made with sands in a lower range of grain sizes than that covered by the basic mixtures. An examination of their curve, Fig. 19, reveals the fact that their suggested lower limit of usefulness for the critical tractive force curve, at an abscissa value of 5.0 (English units), almost exactly coincides with the plotted value of Sand II, the smallest of the three sands tested. Moreover, all the eight highest rated points in Comparison Class (1) (see, Points I, II, III, *A, M, N, P,* and *Q,* in Fig. 14), which formed the principal basis for the critical tractive force curve, fall to the right of their suggested lower limit. In the range of fine mixtures the lower limitation advocated by Messrs. Tiffany and Bentzel is undoubtedly indicated. Such mixtures, however, are inherently unsuited for use in ordinary fluvial models because of their excessive riffling tendency.

The investigations of voids ratio made by Messrs. Tiffany and Bentzel demonstrate the fact that a refinement in the *M*-value is necessary before a general formula for critical tractive force can be developed. It is believed, however, that the basis for this refinement lies in the gradation curve of the sand mixture, and that a definite relationship between the gradation curve and the voids ratio can be discovered. With sufficient experimental data, the mathematical method suggested in the original paper, under the heading, "Practical Derivation of a New Formula: 1.—Procedure,"[62] should reveal this relationship. A more complete explanation of this procedure is contained in the unabridged translation.[57]

Professor Straub has presented some valuable supplementary information, covering a range of tractive force higher than that covered by the basic experiments. It is interesting to note that he has adopted the experimental technique used by Gilbert,[2a] wherein the discharge and rate of sand feeding are held constant, and the experiment is continued until an equilibrium of slope and depth of flow are obtained. Most experimenters, having found this equilibrium difficult to attain and difficult to recognize when it is reached, have preferred to hold the slope and discharge constant, and to adjust the rate of sand feeding to the rate at which the material is being caught in the sand-trap.

Although the knowledge of "geschiebe" movement is still inadequate, extensive research programs now (1935) being undertaken at the U. S. Waterways Experiment Station; at the Prussian Experiment Institute for Hydraulic Engineering and Shipbuilding, by Capt. Hugh J. Casey, Corps of Engineers,

[62] *Proceedings,* Am. Soc. C. E., April, 1934, p. 475.

[2a] "The Transportation of Débris by Running Water," by G. K. Gilbert, *Professional Paper No. 86,* U. S. Geological Survey, Washington, 1914.

U. S. Army, M. Am. Soc. C. E.; at the University of Minnesota by Professor Straub; and at numerous other hydraulic laboratories, should result in definite progress in this science.

Acknowledgment.—These studies have been made more valuable because of the writer's privilege of maintaining a close personal contact with the staff of the U. S. Waterways Experiment Station, where investigations of a similar nature have been carried forward. Acknowledgment is due to Lieut. H. D. Vogel, Corps of Engineers, U. S. Army, Assoc. M. Am. Soc. C. E., former Director of the Station, to Lieut. F. H. Falkner, Corps of Engineers, U. S. Army, Jun. Am. Soc. C. E., the present Director, and to Joseph B. Tiffany, Jr., Jun. Am. Soc. C. E., Junior Engineer at the Station, who is in charge of "geschiebe" experiments there, and who has aided in the preparation of both the original paper and the closing discussion.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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DISCUSSIONS

LABORATORY TESTS OF MULTIPLE-SPAN REINFORCED CONCRETE ARCH BRIDGES

Discussion

BY WILBUR M. WILSON, M. AM. SOC. C. E.

WILBUR M. WILSON,[13] M. AM. SOC. C. E. (by letter).[13a]—In the absence of any considerable disagreement with the interpretation of the data, little appears to be needed in the way of a closure.

Mr. McCullough has mentioned the desirability of a more complete presentation of the combined algebraic-and-experimental method of analyzing multiple-span arch series. This method is presented in the Final Report of the Special Committee on Concrete and Reinforced Concrete Arches.[14]

Quite properly, Mr. McCullough calls attention to the possibility of an increase in stress at the springing of one arch due to live load on adjacent arches. The live load used in the tests is for highway bridges and the single heavy concentration overshadows the smaller distributed load. For a railway bridge, the load on spans adjacent to the one being considered might produce considerable stress thereby increasing the difference between the stress on a multiple-span structure and that in a similar single span. A point that the writer neglected to mention, is the fact that, although the negative live load moment at the springing of the middle span is reduced by the flexure of the pier, the positive moment at that section and the negative moment at the abutments are increased.

Mr. Mann's analysis of a three-span arch series by means of an elastic model is of interest, showing as it does the agreement that may be obtained between values derived from tests of models and from an algebraic analysis when sufficient experience has been acquired to assure skill and accuracy with both methods. However, the experience of the writer is that for a three-span structure consisting of ribs without deck, which are readily susceptible of algebraic analysis, less time is required and the results are more acceptable if the structure is analyzed by the elastic theory than by the use of an elastic model. In the analysis of a three-span structure with

NOTE.—The paper by Wilbur M. Wilson, M. Am. Soc. C. E., was presented at the Joint Meeting of the Structural Division, Am. Soc. C. E., and the Applied Mechanics Division, Am. Soc. M. E., Chicago, Ill., June 29, 1933, and was published in April, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: August, 1934, by C. B. McCullough, M. Am. Soc. C. E.; September, 1934, by Carroll L. Mann, Jr., Esq.; and December, 1934, by M. Hirschthal, M. Am. Soc. C. E.

[13] Research Prof. of Structural Eng., Univ. of Illinois, Urbana, Ill.

[13a] Received by the Secretary, March 23, 1935.

[14] Publication pending.

a deck, the spans of which are not readily susceptible of algebraic analysis, an experimental analysis involving the use of elastic models appears to be the most acceptable; and comparisons such as Mr. Mann and others have made would appear to indicate that the results of such an analysis are dependable, providing the tests are properly planned and care is exercised to prevent experimentation errors. Whether the model analysis is better accomplished by tests of a model of the entire structure or by tests of a model of a single span, using the experimental data in the combined algebraic-and-experimental method is, to a certain extent, a matter of personal preference. The writer prefers the latter, and for the following reasons.

The difference in the time required to make the model—between a single-span and a three-span model—is more than that required to make the algebraic analysis. The results of the algebraic analysis are statically consistent, which is not always true with the results of an all-experimental method. With the combined algebraic-and-experimental method a considerable error in the experimental constants produces only a small error in the stress in the rib computed from the constants.

The three-span structure with the low deck, shown in Fig. 2(g), was analyzed by the use of an elastic model, and the results agree closely with those obtained from tests of the concrete specimens.[15] The difference between the results obtained by the two methods are possibly no greater than the differences that might exist between the behavior of two concrete arch bridges built to the same drawings and specifications.

Mr. Hirschthal directs attention to the influence upon the rib of the action of the spandrel columns and deck, and particularly the influence of expansion joints in the deck. These are important questions and questions for which there are as yet no definite and final answers. The early impression was that the interaction of the rib and deck was an asset since it appeared to increase the load-carrying capacity of the structure. More recently attention has been directed to the detrimental effect of the interaction which, although increasing the load-carrying capacity of the rib, may overstress the spandrel columns and deck. Possibly, the final solution (particularly for extremely long-span arches) will be to develop a design having such a relation of rib to superstructure that it will be possibe to take advantage of the increase in the load-carrying capacity resulting from interaction without injuriously overstressing the spandrel columns and deck. This is desirable, not so much because of the direct saving in material, but because of the reduction in the dead load, which constitutes almost the entire load in a long-span, reinforced concrete arch bridge. This is an important hoped for development of the future. If the tests, which are the subject of the paper, have contributed to this solution, or if they have demonstrated the desirability of such a design, the investigation will have been profitable.

Mr. Hirschthal relates his experiences with the concrete arches of the Delaware, Lackawanna, and Western Railroad Company and states that cracks occurred in the parapets of two bridges built without intermediate expansion joints and did not occur in the parapets of two other bridges having intermediate expansion joints.

[15] See *Bulletin 270*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill.

The writer would distinguish between expansion joints in the deck and in the parapet. The horizontal movement that would occur if intermediate expansion joints were provided increases with the distance above the rib. The top of the parapet is a considerable distance above the rib at the crown for these bridges. The cracking of the parapet could have been prevented by providing intermediate expansion joints in it whether or not there were corresponding expansion joints in the deck.

If the arch barrel, or rib, for an open spandrel arch is extremely heavy so as to overwhelm the deck, as it was for the Delaware River Bridge,[16] referred to by Mr. Hirschthal, the stiffer the deck the more likely it is to crack if intermediate expansion joints are not provided. Mr. Hirschthal kindly furnished the writer with blue-prints showing the details of the bridges referred to by him. The deck for all but the Delaware River Bridge was a considerable distance above the rib at the crown, and the bridges fall in the class for which the tests by the writer[17] show that a considerable horizontal movement would occur if intermediate expansion joints were provided. The deck for the Delaware River Bridge was high enough to be not integral with rib at the crown although not so high as for the other bridges. Without having sufficient knowledge as to the interaction between the rib and deck to form a final opinion, it appears to the writer that there are features of this bridge that make it particularly liable to crack if intermediate expansion joints are not provided. Although the deck is not especially high, the parapet is high enough so that some horizontal movement would have taken place at intermediate expansion joints if they had been provided. The arch barrel is extremely heavy. The deck spans are short and, therefore, so stiff that they cannot conform to the flexure of the rib without cracking and yet are not strong enough to prevent this flexure. The situation is further aggravated by the fact that the section of the deck is less at the center of the deck span than at the transverse spandrel walls.

It is possible that the need for intermediate expansion joints in the deck of an arch bridge is not entirely a question of the position of the deck relative to the rib, although that is a vital relation, and both tests and a visualization of the action of the structure indicate that the horizontal movement that would occur if expansion joints were provided is a function of the height of the deck above the rib. The relative stiffness of the deck, spandrel columns, and rib is also important as they probably affect the vertical and angular motion that would occur if expansion joints were provided; they also affect the stress that will result if the movement is prevented or reduced by making the deck continuous.

The studies made to determine the movement at expansion joints[17] were limited to horizontal movement, the opening and closing of the joint. In view of the questions raised by Mr. Hirschthal, further studies to determine the relative vertical and angular movement of the two ends of the deck that meet at an expansion joint, appear to be desirable. This can be done in an acceptable manner with elastic models by methods similar to those used in studying the horizontal movement.

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[16] *Engineering News*, Vol. 62, No. 26, p. 713.

[17] *Bulletin 226* and *Bulletin 270*, Univ. of Illinois Eng. Experiment Station.

Another question pertaining to the design of multiple-span arch bridges which the writer believes should obtain more attention than it receives, at least in the literature on arches, is the distribution of the vertical distance from the bases of the piers to the top of the roadway. The elevation of the roadway is usually fixed by the approaches, and that of the bases of the piers by the foundation conditions. The difference between these two elevations is usually fixed within narrow limits, but the total vertical distance can be divided into three parts at the discretion of the designer. These parts are: (1) The height of the pier; (2) the rise of the arch; and (3) the distance of the deck above the rib at the crown.

In so far as the limited number of tests show, for a particular rib and a particular deck, the height of the deck above the rib does not affect the load-carrying capacity of the structure, although it does affect somewhat the stresses resulting from temperature changes. Increasing the height of the deck above the crown of the rib also increases the horizontal movement that would take place at intermediate expansion joints if they are provided. This is compensated for, at least in part, by the greater flexibility of the longer spandrel columns. Therefore, it would appear that, structurally, the high deck has no appreciable advantage over the low deck, or the reverse.

If the total available vertical distance is small, the natural design is to make the piers as low as the required flood channel will permit and make the deck so low as to be integral with the rib at the crown, thereby obtaining a maximum rise for the arch rib. If the total vertical distance is ample, or possibly excessive, the best distribution is not so easily apparent.

Increasing the flexibility of the piers of a multiple-span series increases the live-load stress in the rib, but a considerable increase in the flexibility causes a relatively small increase in the stress. Moreover, increasing the flexibility of the piers has a lesser effect on a rib with a large, than on one with a small, rise ratio. Increasing the height of the pier increases the flood channel. Increasing the rise ratio of the arch decreases the horizontal thrust on the piers due to load and decreases the temperature stress in the rib. It would appear, therefore, that there is no advantage in separating the deck from the rib at the crown as the vertical distance can be better disposed of by increasing the height of the pier and increasing the rise ratio of the arch rib. The use of a multiple-span series of flat arches on high piers would appear to be undesirable.

Mr. Hirschthal emphasizes the care necessary in drawing conclusions from tests of a small structure. The writer wishes to repeat this caution not only in applying the results obtained from a small structure to the design of a large structure, but also in drawing any general conclusions from a limited number of tests. Although the investigation included several tests, the results have little statistical value. In general, they are of value because they indicate how the structure functions rather than because of the absolute quantitative values obtained. Those interested are urged to study the detailed reports[18] and make their own interpretation of the data rather than depend entirely upon the writer's interpretation.

[18] *Bulletin 269* and *Bulletin 270*, Univ. of Illinois Eng. Experiment Station.

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DISCUSSIONS

ECCENTRIC RIVETED CONNECTIONS

Discussion

By Messrs. Fang-Yin Tsai, Odd Albert, and Eugene A. Dubin

Fang-Yin Tsai,[16] Assoc. M. Am. Soc. C. E. (by letter).[16a]—The formulas and alignment charts presented in this paper are certainly great improvements over similar devices for the solution of the problem, and the author is to be commended for his service to the Engineering Profession. Four alignment charts are presented for the four common cases (group of rivets consisting of one, two, three, and four vertical lines), in which the vertical pitch, p, is fixed at 3 in. uniformly. Although a uniform 3-in. pitch is usually to be preferred in structural detailing, this is not always possible, and, in the worst case, a varying pitch may have to be used to fit some unusual condition. If a uniform pitch, p, is also taken as a variable within a certain range (say, from $2\frac{1}{2}$ in. to 6 in., with a variation of $\frac{1}{4}$ in.) in constructing those charts, they would certainly have a much wider range of application.

Four sets of formulas are also presented for the four common cases. These formulas are not only too cumbersome, but too limited in their application, because no equation is given for any rivet group consisting of more than four vertical lines. Elsewhere, the writer has derived[17] a general formula that may be applied to a rivet group consisting of any number of vertical lines, with the limitations that all vertical lines have the same number of rivets and the same uniform pitch, and that the line of action of the eccentric load is parallel to the vertical line of rivets.

Although the formulas therein presented are somewhat lengthy they are general and comprehensive and are preferable to shorter formulas applicable to particular cases only. Alignment charts constructed on the basis of the writer's more general formulas would undoubtedly have a much wider application than those presented by the author.

Note.—The paper by Eugene A. Dubin, Esq., was published in August, 1934, *Proceedings*. Discussion on this paper has been published in *Proceedings*, as follows: December, 1934, by Messrs. Carlton T. Bishop, James R. Bole, Albert Wertheimer, Kenneth L. DeBlois, Jonathan Jones, Armin Rozman, and A. E. R. de Jonge.

[16] Prof. of Structural Eng., National Tsing Hua Univ., Peiping, China.

[16a] Received by the Secretary November 3, 1934.

[17] *Civil Engineering* March, 1935, p. 178.

ODD ALBERT,[18] ASSOC. M. AM. SOC. C. E. (by letter).[18a]—Credit is due Mr. Dubin for the novel manner in which he presents his problems. Unfortunately, the method advanced in the paper will not allow the use of more complicated formulas, because as the author modestly admits in connection with Equation (10) "the resulting function cannot be transformed to fit any standard form for which an alignment chart may be plotted."

By using the so-called "successive charts" it is possible, however, to express graphically almost any kind of formulas; but it is most important to have these formulas first transformed into a certain type that can be used in plotting a chart. Very often this is a problem in itself. As a case in point, consider the following formula:[19]

$$s = \frac{P}{n} \sqrt{C^2 B + C + 1} = \frac{P}{n} e \quad \text{(29)}$$

in which, in addition to the notation of the paper,

$$C = \frac{e\,n\,w}{\Sigma\, r^2} = \frac{12\,e\,w\,(N-1)}{p^2\,(n^2_o - 1) + w^2\,(N^2 - 1)} \quad \text{(30)}$$

and,

$$B = \frac{r^2_o}{w^2} = \frac{p^2\,(n_o - 1)^2 + w^2\,(R-1)^2}{4\,w^2\,(N-1)^2} \quad \text{(31)}$$

In Equations (30) and (31) n_o = number of rivets in each row, and N = number of vertical rivet rows. Equation (29) corresponds with Equation (8) of the paper, except that it is more general and applies to any number of rivet rows.

If applied to two rows of rivets, Equations (30) and (31) take the form:

$$C = \frac{12\,e\,w}{p^2\,(n^2_o - 1) + 3\,w^2} \quad \text{(32)}$$

and,

$$B = \frac{p^2\,(n_o - 1)^2 + w^2}{4\,w^2} \quad \text{(33)}$$

The adaptation of Equation (29) to graphical solution may be demonstrated by reference to Fig. 9, to determine the maximum rivet stress for the extreme rivet in an eccentrically loaded rivet group under the condition that the vertical pitch, p, and the horizontal spacing, w, are equal. For this case,

$$C = \frac{12\,e}{w\,(n^2_o + 2)} \quad \text{(34)}$$

and,

$$B = \frac{(n_o - 1)^2 + 1}{4} \quad \text{(35)}$$

[18] Faculty Lecturer, Advanced Design of Structures, New York Univ., New York, N. Y.

[18a] Received by the Secretary April 17, 1935.

[19] *Civil Engineering*, February, 1935, p. 414, Equation [2].

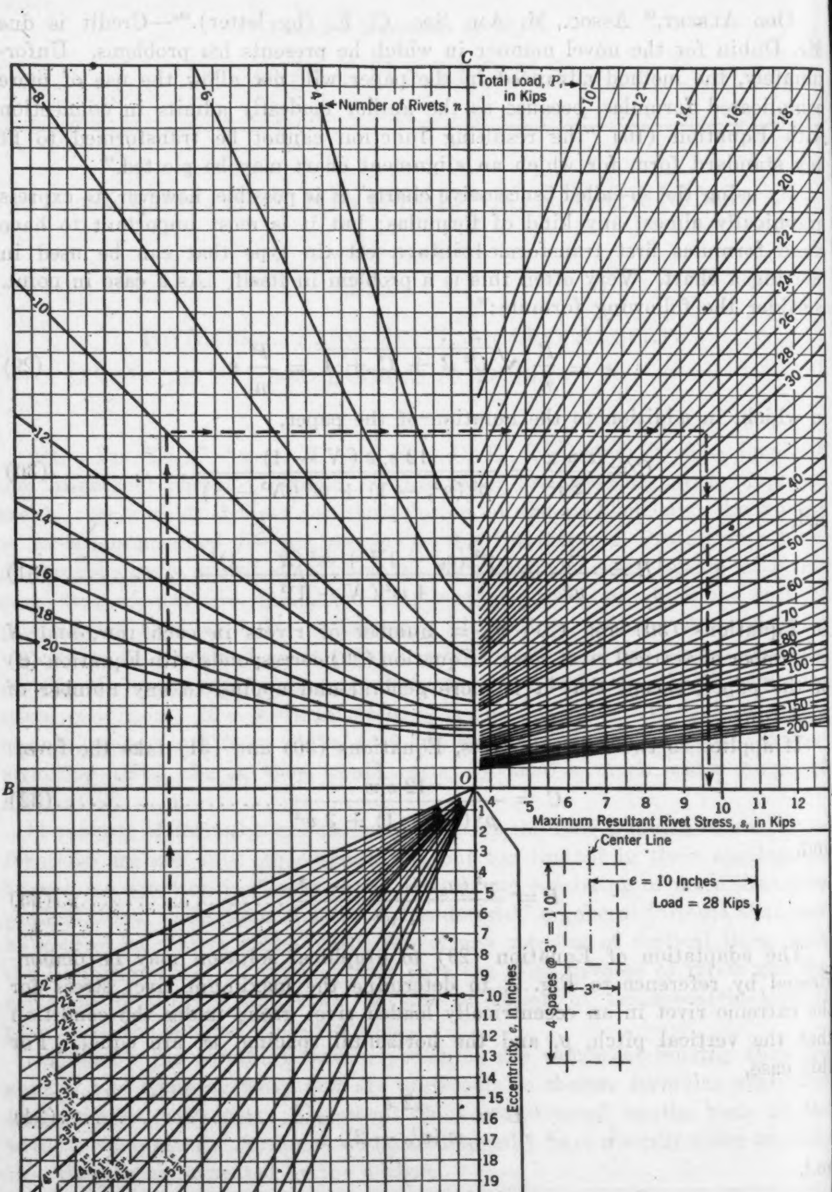

FIG. 9—ECCENTRIC RIVET CONNECTIONS WITH TWO VERTICAL ROWS OF RIVETS.

Consider a connection to be designed with ten rivets in two vertical rows. The load, $P = 28\,000$ lb (28 kips), applied with an eccentricity of 10 in. Let $w = p = 3$ in. In Fig. 9, beginning at $e = 10$ in., intersect $p = w = 3$, horizontally. From this intersection produce a vertical line to an intersection with $n = 10$. A horizontal line from this latter intersection will cut $P = 28$ such that the final vertical line yields $s = 9\,650$ lb.

For the case treated in Example 2 of the paper, Equation (29) may be used again. From Equations (30) and (31):

$$C = \frac{12\,e\,w}{9\,(n_o^2 - 1) + 3\,w^2} \quad \text{(36)}$$

and,

$$B = \frac{9\,(n_o - 1)^2 + w^2}{4\,w^2} \quad \text{(37)}$$

In this connection it may be mentioned that the polar moment of inertia, Σr_o can be expressed as follows:

$$\Sigma\, r_o = \frac{[(n_o^2 - 1)\,p^2 + 3\,w^2]\,n}{12} \quad \text{(38)}$$

which is a simpler expression than Equation (10). For a rivet group of N rows of rivets the following expression may be derived:

$$\Sigma\, r_o = \frac{[(n_o^2 - 1)\,p^2 + (N^2 - 1)\,w^2]\,n}{12} = \frac{e\,n\,w\,(N - 1)}{C} \quad \text{(39)}$$

Equation (39) is easily converted into the forms of Equations (16) and (22) by making $N = 3$ and 4, respectively, and $n_o = \frac{n}{3}$ and $\frac{n}{4}$, respectively.

For the distance, r_o:

$$r_o = w\,(N - 1)\,\sqrt{B} \quad \text{(40)}$$

and, for two vertical rivet rows, for example:

$$r_o = w\,\sqrt{B} = \tfrac{1}{2}\,\sqrt{p^2\,(n_o - 1)^2 + w^2} \quad \text{(41)}$$

It will be seen that Equation (40) is very simple as compared to Equations (9) and (18), and has a further advantage in being applicable to any number of vertical rivet rows, as has been shown by Equation (41).

If a rivet group is subject to bending only, the maximum rivet stress will be,

$$s = \frac{P}{n}\,C\,\sqrt{B} \quad \text{(42)}$$

in which the values of C and B refer to Equations (30) and (31).

Thus far, the methods presented and charts described are mathematically correct, with no approximations. Mr. Dubin states that the results determined by using his charts may be in error as much as 5 per cent. If accuracy may be sacrificed, somewhat, in the interest of speed, it should be possible to save considerable time by assuming that: The rivets are in one, two, or

more, vertical rows, symmetrically arranged in respect to a horizontal axis; in every row, pairs of rivets at the same distance from and on opposite sides of the neutral axis form couples and take moments in proportion to the distance between them; and the spacing between them in both directions equals three to five times the rivet diameter. Then, for one vertical row of rivets, the stress, s_B, on the extreme rivet due to the bending alone is,

$$s_B = \frac{6\,P\,e}{p\,n\,(n+1)} = \frac{P\,e}{p K} \qquad \text{(43)}$$

in which the values of K for given numbers of rivets in one vertical row are as follows:

Number of rivets in each row	Corresponding value of K	Number of rivets in each row	Corresponding value of K
4	$3\frac{1}{3}$	10	$18\frac{1}{3}$
5	5	11	22
6	7	12	26
7	$9\frac{1}{3}$	13	$30\frac{1}{3}$
8	12	14	35
9	15	15	40

If there are several rows of rivets, the sum of the moments that each row is capable of withstanding, is assumed to be equal to the total outside moment. Thus, for instance, in the case of three rows of rivets the moment equals:

For the first row,

$$M_1 = p\,s_{B1}\,K_1$$

for the second row,

$$M_2 = p\,s_{B2}\,K_2$$

and, for the third row,

$$M_3 = p\,s_{B3}\,K_3$$

If the extreme rivets are at the same distance from the neutral axis, the s-stresses are alike. Furthermore, the outside moment equals $M_1 + M_2 + M_3$; and, consequently,

$$M = p\,s_B\,(K_1 + K_2 + K_3) \qquad \text{(44)}$$

and the maximum rivet stress caused by bending alone will be,

$$s_B = \frac{P\,e}{p\,\Sigma K} \qquad \text{(45)}$$

in which ΣK equals the sum of as many terms as there are vertical rivet rows. The number of rivets in each row only determines the value of K for this row.

The Resultant Stress.—As an eccentric load always produces a shear stress, s_A (due to the load divided by the number of rivets), in addition to the s_B-stress, the resultant stress will be,

$$s = \sqrt{s_A^2 + s_B^2} \qquad \text{(46)}$$

which corresponds to Equations (1), (8), and (11). The s_A-stress is always vertical, and the s_B-stress is assumed to be horizontal. As soon as there are two or more rows, this is not true. As the last terms shown in Equations (8), (11), and (14) are omitted, the error caused by this omission is partly offset by the assumption that the s_B-stress is horizontal.

As an example, consider a rivet group, consisting of four vertical rows with thirteen rivets in each row. A vertical load of 100 000 lb is applied with an eccentricity of 32.7 in. If the vertical and the horizontal spacing is $4\frac{7}{16}$ in., the maximum rivet stress is determined as follows: The stress due to shear, $s_A = \frac{100\,000}{52} = 1\,922$ lb, and the value of K (from the list of K-values) $= 30\frac{1}{3}$; therefore, for four rows, $\Sigma K = 4 \times 30\frac{1}{3} = 121.32$; and $s_B = \frac{100\,000 \times 32.7}{4.48 \times 121.32}$ $= 6\,020$ lb. Therefore, the resultant rivet stress will be, $s = \sqrt{1\,922^2 + 6\,020^2}$ $= 6\,310$ lb. The maximum stress should have been 6 490 lb, indicating an error of only 2.85 per cent.

The foregoing approximation is correct for one row of rivets and is accurate well within 5% for two or more rows. Allowing the same tolerance, it is quicker than that proposed by Mr. Dubin.

EUGENE A. DUBIN,[20] ESQ. (by letter).[20a]—It is almost axiomatic that the best structural member is one that has the least number of component parts. The most simple and direct design is always to be preferred. Nevertheless, certain structural details of a complicated nature are necessary. A riveted connection is one such necessary detail.

Mr. de Jonge has indicated the nature of the problem by enumerating the assumptions upon which the paper is based. It is correct that these assumptions do not all hold true in practice; yet the nature of the action of a riveted connection is such that the simplifying assumptions used seem justified.

In any riveted connection, whether the load is eccentric or concentric, the direct stress is not distributed uniformly between the rivets. The distribution of the load among the rivets of a connection of many rows of rivets is quite indeterminate, since the load on one row of rivets depends on the yielding of the rivets in the rows nearer the load, and on the yielding of the plates. In a connection of many rows of rivets, the outermost rivets initially carry more of the direct load than those farther from the load. Over-stress in these rivets causes local yielding of the plates and the rivets, and tends to distribute the load more uniformly among the rivets. In connections of a ductile material, this is not serious. It must be borne in mind that abrupt changes in section of any member, introducing holes in a member (rivet holes), etc., cause localized stresses in the member at the hole, change in section, etc., that may reach considerable magnitude.

[20] Washington, D. C.

[20a] Received by the Secretary April 15, 1935.

The material around rivet holes is seriously damaged by punching. Rivet holes do not match perfectly. Rivets cannot be driven uniformly, especially in the field. Field rivets may not be heated uniformly. These factors all influence the actual distribution of the load.

Mr. de Jonge states,

"* * * the rivet shanks do not bear against the plate at all, at least not if the connection has not been over-strained previously. This is due to the fact that, at working loads, the riveted joint carries the load by the frictional resistance developed between the plates."

A well-driven rivet is assumed to fill the hole. The contraction of the rivet during cooling does introduce clamping action on the plates which develops frictional resistance. Tests have shown,[21] however, that slip may occur at ordinary working stresses.

It is well also to consider the nature of the stress in the rivets. The contraction of the rivet while cooling introduces an indeterminate degree of tension; the rivet is subject to bending (which may have an important influence on long rivets); and it is subjected to a shearing force. A determination of the actual resultant stress in the rivet is impossible. Obviously, neither the shearing stress nor the bearing stress (which is not uniform over the length of the rivet in bearing), is actually the governing stress. However, by tests, empirical values for the allowable shear and bearing have been determined, which offer criteria that make it possible to design riveted connections safely, considering only these stresses. The method presented by the writer offers a solution based upon these criteria. The nature of the problem warrants Professor Bishop's warning, "that absurd results may be obtained when it is applied to extreme and impractical problems."

Mr. de Jonge states that "alignment charts are simple and accurate only when they consist of straight lines." The writer wishes to point out that an alignment chart does not necessarily consist of straight lines. In fact, an exact, mathematically constructed chart for some equations may give one scale that is a curve resembling the letter, *S*, etc. The method of constructing alignment charts is well presented in standard works on the subject.[6]

The charts originally presented were restricted to groups of rivets with a 3-in. pitch, and, in Examples 3 and 4, to a spacing between rows of $w = 3$ in. Professor Bishop, Professor Tsai, and Mr. de Jonge have suggested that the charts would be more useful if they were extended to cases in which the pitch differed from 3 in. Mr. John Krippner suggested to the writer that the ratio, $\frac{e}{p}$, instead of e be plotted on one scale. By means of this device the charts can be used for any equal vertical spacing of rivets.

[21] "Tests of Nickel-Steel Riveted Joints," by Arthur N. Talbot, Past-President and Hon. M., Am. Soc. C. E., and Herbert F. Moore, *Bulletin No. 49*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

[6] The method developed by Mr. Albert Wertheimer, which was used to construct the charts for Examples 2 to 4, inclusive, has been published in the *Journal* of the Franklin Institute, Vol. 219, No. 3, March, 1935, pp. 343–363.

Consider Example 1 of the paper, in which $\frac{e}{p} = m$. For this case, Equation (4) becomes,

$$\left(\frac{1}{A}\right)^2 - m^2 \left[\frac{6}{n\,(n+1)}\right]^2 = \left(\frac{1}{n}\right)^2 \quad \text{.................} (47)$$

Since $p = 3$ in. was used to plot Fig. 1, this chart may be revised by dividing the numerical values of the "eccentricity" scale by 3, and marking it "Values of m" instead of "Eccentricity, e, in inches."

Mr. Rozman states that Fig. 1 cannot have any use in designing plate-girder web splices, since a chart for such purposes requires that the range of the eccentricity be "about 15 ft to infinity." Fig. 1 does cover only a limited range, but Mr. Rozman's statement is in error. The moment in the web is not the total moment in the girder. It is common to assume that one-eighth the area of the web acts as flange material. The moment in the web is, $M_w = \dfrac{\frac{1}{8} A_w}{A_f + \frac{1}{8} A_w} \times M_G$, in which M_w = moment in the web; A_w = area of the web; A_f = area of the flange; and, M_G = total moment in the girder. The moment in the web approximates 5% to 15% of M_G.

Assuming that $M_w = 0.1\ M_G$, consider Mr. Rozman's examples. In Fig. 8(a), assume P at the mid-span, L, varying from 30 ft to 100 ft, and

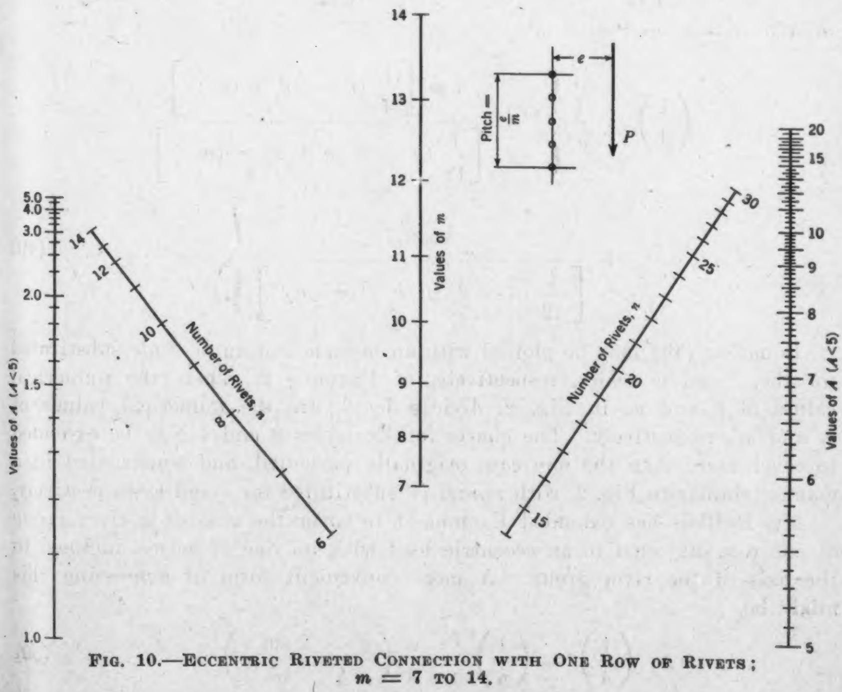

FIG. 10.—ECCENTRIC RIVETED CONNECTION WITH ONE ROW OF RIVETS; $m = 7$ TO 14.

web splices at $\frac{L}{3}$. At the web splice, the moment in the web due to P is $\frac{PL}{4} \times \frac{2}{3} \times \frac{1}{10} = \frac{PL}{60}$, and the shear due to P is $\frac{P}{2}$. The eccentricity then is $\frac{L}{30}$, or 1 ft to 3.33 ft for the range in spans considered. In Fig. 8(*b*), with a uniform load per foot, W, for the same range in spans and splices at the same points,

$$e = \frac{WL^2}{9} \times \frac{1}{10} \div \frac{WL}{6} = \frac{L}{15}$$

or 2 ft to 6.66 ft for the range of spans considered. Fig. 1 of the paper did not cover this range; therefore, Fig. 10, similar to Fig. 1, is extended to $m = 14$, which is equivalent to $e = 42$ in. when $p = 3$ in.

The charts for Examples 2, 3, and 4 of the paper may be revised to provide for any equal vertical rivet spacing and any distance between rivet rows in the manner used for Example 1. For example, consider Example 2; substituting the values of r_0 and $\Sigma\, r^2$ in Equation (11):

$$\left(\frac{1}{A}\right)^2 = \frac{1}{n^2}\left\{1 + \frac{4e^2\left[\frac{p^2}{4}(n-2)^2 + w^2\right]}{\left[\frac{p^2}{12}(n-2)(n+2) + w^2\right]^2} + \frac{4\,ew}{\left[\frac{p^2}{12}(n-2)(n+2) + w^2\right]}\right\} \quad \text{..(48)}$$

or, if $e = mp$; and $w = m'p$:

$$\left(\frac{1}{A}\right)^2 = \frac{1}{n^2}\left\{1 + \frac{4\,m^2\left[\frac{1}{4}(n-2)^2 + (m')^2\right]}{\left[\frac{1}{12}(n-2)(n+2) + (m')^2\right]^2} + \frac{4\,m'\,m}{\left[\frac{1}{12}(n-2)(n+2) + (m')^2\right]}\right\} \quad \text{..............(49)}$$

Equation (49) may be plotted with an m-scale and an m'-scale substituted for the e and w scales, respectively, of Example 2. Then, the numerical values of e and w, in Fig. 2, divided by 3, are the numerical values of m and m', respectively. The charts for Examples 3 and 4 may be extended to cover more than the one case originally presented, and constructed in a manner similar to Fig. 2, with m and m' substituted for e and w, respectively.

Mr. DeBlois has extended Example 1 to cover the case of a rivet group of one row subjected to an eccentric load with its line of action inclined to the axis of the rivet group. A more convenient form of expressing this might be,

$$\left(\frac{1}{A}\right)^2 = \left(\frac{1}{n}\right)^2 + \frac{e}{S}\left(\frac{e}{S} + \frac{2\sin\alpha}{n}\right) \quad \text{.............(50)}$$

in which α is the angle between the axis of the rivet group and the action line of the load (see Fig. 11).

Mr. Rozman has developed a formula for the solution of bracketed connections. His assumption that the neutral axis of the tension and compression areas is the center of gravity of the rivet group, although commonly

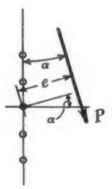

FIG. 11.

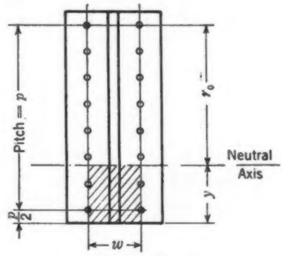

FIG. 12.

made, is not correct, theoretically. As he states, however, the assumption is on the side of safety. It must be borne in mind, nevertheless, that the tension in the rivets is normal to the plane upon which the shear occurs, and, therefore, Equation (28) does not hold. A bracketed connection must satisfy three requirements: (1) The bearing of the rivets must not exceed the allowable value of the bearing for the direct load; (2) the maximum shear on the rivet shank due to the direct load and the tension must not exceed the allowable shear value of the rivet; and (3) the maximum tension in the rivet due to the shear and the tension caused by the moment, must not exceed the allowable value of the rivet in tension. The design or investigation of a bracket for bearing on the rivets due to the direct load needs no discussion.

The maximum unit shearing stress on a member subjected to normal tension and direct shear is,

$$s'_s = \tfrac{1}{2} \sqrt{s_t^2 + 4\, s_s^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

in which s_t = the tensile unit stress; and, s_s = the shearing unit stress. The maximum unit tensile stress is,

$$s'_t = \tfrac{1}{2}\, s_t + \tfrac{1}{2} \sqrt{s_t^2 + 4\, s_s^2} \quad \ldots\ldots\ldots\ldots\ldots (52)$$

Since, for steel rivets the allowable tensile unit stress is equal to the allowable shearing unit stress, only the maximum tension need be considered when the rivets are of steel. In the bracketed connection shown in Fig. (7a),

$$s_t = \frac{P e r_0}{I\, a_r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

in which r_0 = the distance from the neutral axis of the rivets in tension and the area in bearing, to the topmost rivet; a_r = the area of a rivet; and

I = the moment of inertia of the rivets in tension and the area in bearing about the neutral axis. Similarly,

$$s_s = \frac{P}{a_r\, n} \quad \text{......................... (54)}$$

and, $s'_t\, a_r = \frac{1}{2} \times \frac{Pe r_0}{I} + \frac{1}{2}\sqrt{\left(\frac{Pe r_0}{I}\right)^2 + 4\left(\frac{P}{n}\right)^2}$; from which,

$$\frac{2}{A} = \frac{e}{S} + \sqrt{\left(\frac{e}{S}\right)^2 + 4\left(\frac{1}{n}\right)^2} \quad \text{................ (55)}$$

It is possible to determine the location of the neutral axis with a reasonable degree of accuracy. Assume that the width of angles between the rivets is in bearing, and that the distance from the bottom of the angles to the bottom horizontal row of rivets is $\frac{p}{2}$. Assume, also, that the neutral axis lies midway between two horizontal rows of rivets (see Fig. 12). The statical moments of the tension and compression areas about the neutral axis are equal, and,

$$\frac{wy^2}{2} = 2\, a_r\, \Sigma\, r \quad \text{...................... (56)}$$

in which y (the distance from the neutral axis to the bottom of the angles) $= n_2 p$; $\Sigma\, r = \frac{p\, n_1^2}{2}$; n_2 = the number of rivets below the neutral axis in one row; and n_1 = the number of rivets above the neutral axis in one row. Therefore,

$$n_2 = n_1 \sqrt{\frac{2\, a_r}{w\, p}} \quad \text{...................... (57)}$$

or, $n_2 = \frac{1}{2}\, n - n_1$; and,

$$n_1 = \frac{n}{2\left(1 + \sqrt{\frac{2\, a_r}{w\, p}}\right)} \quad \text{...................... (58)}$$

Neglecting the moment of inertia of the rivet areas about their neutral axis, $I = 2a_r\, \Sigma\, r^2 + \frac{wy^3}{12} + wy\left(\frac{y}{2}\right)^2$; or:

$$I = \frac{a_r\, p^2\, n_1}{6}\left[(2\, n_1 + 1)\,(2\, n_1 - 1) + 4\, n_1^2 \sqrt{\frac{2\, a_r}{w p}}\right] \quad \text{........ (59)}$$

Furthermore, $r_0 = \frac{p}{2}\,(2n_1 - 1)$; and:

$$S = \frac{I}{r_0} = \frac{a_r\, p\, n_1}{3\,(2\, n_1 - 1)}\left[(2\, n_1 + 1)\,(2\, n_1 - 1) + 4\, n_1^2 \sqrt{\frac{2\, a_r}{w p}}\right] \quad \text{... (60)}$$

Equations (58) and (60) may be substituted in Equation (55), and for constant values of a_r, p, and w, an alignment chart may readily be constructed. Thus, the problem does not involve too many unknowns for a practical solution which observes theoretical considerations.

Mr. Bole prefers a family of curves to an alignment chart and Mr. Jones prefers tables. In these matters the individual habits of the engineer govern. Contrary to the opinion of Mr. Bole, however, alignment charts are accurate, and are not difficult to construct.

As Mr. Wertheimer points out, any set of data in three variables may be solved by the use of his method. The method involves a series of successive approximations whereby the results obtained may be brought to any desired degree of accuracy. The charts for Examples 2 to 4, inclusive, are accurate to within 5%, as shown, but they could be brought to a greater degree of accuracy. In view of the nature of the problem, this does not seem justified.

Mr. Albert has extended the problem to a solution similar to that of the writer by the use of a "successive chart." The chart presented in Fig. 9 offers a solution similar to the charts presented in Fig. 2. Fig. 9 is more limited in its scope, however, more cumbersome to use, and no more accurate in results. It is restricted to equal horizontal and vertical spacing of rivets, whereas Fig. 2 may be used for any equal vertical spacing between $m = 0$ and $m = 6\frac{2}{3}$, and any horizontal spacing between $m' = 1$ and $m' = 3\frac{1}{3}$. To solve the problem presented by Mr. Albert, lay a straight-edge across the three scales of Fig. 2(*b*) intersecting e at 10 in. (or $\dfrac{e}{p} = m$ at $3\frac{1}{3}$) and $n = 10$ at $w = 3$ in. (or $\dfrac{w}{p} = m' = 1$). The intersection on the A-scale gives $A = 2.9$; then $s = \dfrac{P}{A} = \dfrac{28\,000}{2.9} = 9\,650$ lb. The result agrees with that obtained by Fig. 9, and the solution was simpler.

Both Professor Tsai[22] and Mr. Albert have extended the problem to cover any number of vertical rows of rivets. The solution of this general case by either an alignment chart or a "successive chart" is not possible without reducing the number of variables involved by considering a specific number of vertical rows of rivets.

To facilitate the solution of an eccentric connection of more than one row of rivets, Mr. Albert proposes the assumption that "pairs of rivets at the same distance from and on opposite sides of the neutral axis form couples and take moments in proportion to the distance between them." He has presented a simplified formula and a table of K-values to aid in solving the formula. The method is the usual "cut-and-try" procedure of assuming the number of rivets first and then checking the connection to see that it is satisfactory. The assumption is justified, however, only when there are many rivets in each vertical row, and the distance between rows is not great.

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[22] *Civil Engineering*, Vol. 5, No. 3, March, 1935, pp. 178–179.

Mr. Albert's proposal is essentially that the stress in each rivet due to bending shall be in proportion to the distance of the rivet from the neutral axis. What Mr. Albert does in his approximate solution is to divide the load uniformly between the number of rows of rivets in the connection and treat it as a connection of one row of rivets. In the introduction to the paper, the writer suggested that, with the foregoing conditions, "Fig. 1 may be used for a preliminary design for connections of more than one row of rivets that lie beyond the range of the other charts." The design is preliminary in character when the number of rivets in each vertical row is not large, or when the distance between vertical rows is great.

Fig. 1 (or Fig. 10, which is Fig. 1 extended) entirely eliminates the necessity for Mr. Albert's formulas and tables. Consider Mr. Albert's problem involving four vertical rows of thirteen rivets, under the assumption proposed herein: $P = 100\,000$ lb; $e = 32.7$ in.; and $p = 4\frac{7}{16}$ in. To find the resultant load on the extreme rivet, lay a straight-edge across the scales of Fig. 10 intersecting m at $\dfrac{e}{p} = \dfrac{32.7}{4\frac{7}{16}} = 7.39$ and $n = 13$. Thus, $\dfrac{P}{s} = 3.75$ and $s = \dfrac{25\,000}{3.75}$, or 6 660 lb. Since the theoretical value of $s = 6\,490$ lb, the error is 2.62 per cent. The error is acceptable. The solution is more readily accomplished than by use of Mr. Albert's tables and formulas.

Mr. Albert states that "the approximation is accurate well within 5% for two or more rows." This is not true, except as outlined previously. Consider the problem of four vertical rows of rivets with two rivets in each row: $p = w = 3$ in; $P = 10\,000$ lb; and $e = 15$ in. From Fig. 4 (*b*), $\dfrac{P}{s} = 1.3$, or $s = 7\,700$ lb. By Mr. Albert's method: $K = \dfrac{n\,(n+1)}{6} = 1$; and, from Equation (45), $S_B = \dfrac{10\,000 \times 15}{3 \times 4 \times 1} = 12\,500$ lb and $S_A = \dfrac{10\,000}{8} = 1\,250$ lb. By solving Equation (46), $S = 12\,580$ lb. Consequently, the error is between 60 and 65 per cent.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

DETERMINATION OF TRAPEZOIDAL PROFILES FOR RETAINING WALLS

Discussion

By A. J. Sutton Pippard, M. Am. Soc. C. E.

A. J. Sutton Pippard,[7] M. Am. Soc. C. E. (by letter).[7a]—The three discussions of this paper were helpful contributions. Mr. Lindley refers to earlier nomograms devised for the same purpose as those given; it is evident that the results of formulas of the Rankine type can be expressed in a variety of ways, and it is not surprising that the nomographic representation has been adopted previously. The publication to which Mr. Lindley refers[4] is not readily accessible, however.

The extensions of Equations (10) and (11) given by Mr. De Blois for the case in which a live load is carried on the fill are simple and useful.

Mr. Drucker points out the apparent anomaly that a wall designed on the Rankine hypotheses is safer when carrying a surcharge that when subjected to the pressure from a level fill. As he mentions, this arises from the fact that the resultant earth thrust is assumed to act parallel to the slope of the fill and is inherent in Rankine's treatment. In the simplest form of the wedge theory the pressure is assumed to act normally to the back of the wall, and walls designed on this basis are too heavy. By introducing consideration of the frictional resistance of the wall the direction of thrust is modified and a nearer approach to the Rankine result is obtained; but the value to be used for the coefficient of friction between earth and wall is a debatable point. Until a complete theory of the behavior of earth is obtained, the design of retaining walls must be largely a matter of judgment; fortunately, workers in many countries are helping toward a much more complete understanding of the intricate problems involved.

Note.—The paper by A. J. Sutton Pippard, M. Am. Soc. C. E., was published in August, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs E. S. Lindley, and Kenneth L. DeBlois; and February, 1935, by M. A. Drucker, Esq.

[7] Prof. of Civ. Eng., Univ. of London, Imperial Coll., City and Guilds (Eng.) Coll., London, England.

[7a] Received by the Secretary April 17, 1935.

[4] "Panjab Irrigation Nomograms," by E. S. Lindley, M. Am. Soc. C. E., Panjab Irrig. Branch, Public Works Dept., Panjab, India.

The writer realized that other considerations, mentioned by Mr. Drucker, must be taken into account in the design of a wall, but once a profile has been obtained which eliminates the possibility of tension on mortar joints it is a simple matter to check the remaining possibilities of failure. In many cases these will be found to be amply safeguarded.

In conclusion the writer would like especially to thank Mr. Drucker for the trouble he has taken in illustrating the points mentioned in his discussion.

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DISCUSSIONS

SECURITY FROM UNDER-SEEPAGE MASONRY DAMS ON EARTH FOUNDATIONS

Discussion

By Messrs. C. H. Eiffert, E. H. Burroughs, Alexander Potter, and Robert E. Kennedy

C. H. Eiffert,[57] M. Am. Soc. C. E. (by letter).[57a]—The writer has always been of the opinion that, in sand and gravel foundations such as those for most low-head dams in the southwestern part of Ohio, the seepage or percolation is more likely to take the short path than to follow the line of creep. The writer agrees absolutely with Mr. Lane in the statement that for clay or hardpan foundations the short-path principle is inferior to the line-of-creep analysis. However, it seems quite possible that in a porous foundation of sand and gravel the line of least resistance would be the short path through this material and, as Mr. Lane states, it seems axiomatic that the line of least resistance will be taken by the seepage.

It seems reasonable to assume that as a rule there will be greater resistance to flow along vertical joints than along those which are horizontal. The weight of the material will cause it to remain in close contact with a vertical row of sheet-piling and, for the same reason, it may draw away from the horizontal surface of the bottom of the dam, at least in places (especially if there is pumping during construction which may draw out the finer material). However, one should not overlook the fact that, although the sheet-piling will have a smooth surface, the concrete bottom of the dam will not be smooth if poured on a gravel foundation; it will be forced down into the voids of the gravel so as to form a more or less interlocking bond which, if settlement does not take place, may offer greater resistance to flow than the contact with the smooth surface of steel sheet-piling or the short path through the material itself.

Note.—The paper by E. W. Lane, M. Am. Soc. C. E., was published in September, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs. William P. Creager, and L. F. Harza; January, 1935, by Messrs. Joel D. Justin, and Louis E. Ayres; February, 1935, by Messr W. M. Griffith, and E. Mckenzie Taylor; and March, 1935, by Messrs. Donald J. Hebert, Arthur Casagrande, and Calvin V. Davis.

[57] Chf. Engr. and Gen. Mgr., The Miami Conservancy Dist., Dayton, Ohio.

[57a] Received by the Secretary February 16, 1935.

In Table 1 (*b*), the Hamilton, Ohio, Dam (No. 33), is listed as having a weighted-creep ratio of 5.5. The material under this dam is a dense, well graded gravel, varying in size from boulders to sand. A few years ago, in making repairs to the paving below the apron a trench about 6 ft deep was dug the entire length of the apron along its down-stream edge. This was unwatered in sections and practically no seepage could be detected coming from beneath the dam. It is possible, therefore, that the length of travel for seepage under this dam is greater than necessary as the conditions in actual use will be much less severe than those imposed during the repair work. However, it is also quite probable that the bed of the stream above the dam has been sealed by silt deposits so that a greater ratio was necessary immediately after construction than is now required.

No mechanical analyses have been made of materials coming from the foundation of this dam, but several such analyses were made of materials taken within a few hundred feet of the dam and which appear to be the same

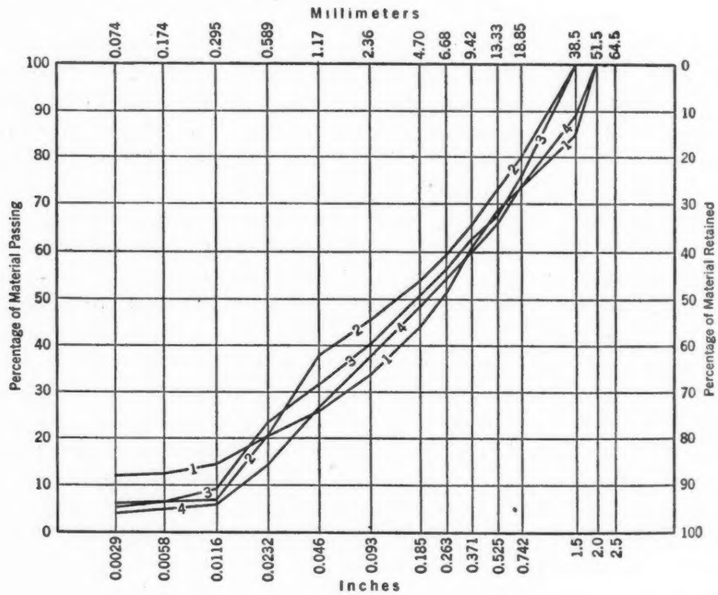

FIG 9.

type of gravel. These are presented in Fig. 9, and although it is not possible to determine the required ratio from them, they do establish the fact that, for this type of material, the existing ratio at this dam is sufficient.

E. H. BURROUGHS,[58] M. AM. SOC. C. E. (by letter).[58a]—The increase in the sum of engineering knowledge as to adequate treatment for porous foundations during the past two or three decades is made apparent by a comparison

[58] Vice-Pres., Ambursen Dam Co., Inc., New York, N. Y.

[58a] Received by the Secretary March 18, 1935.

of Mr. Lane's paper with that presented in 1911, by Mr. Arnold C. Koenig.[59] Reading the author's paper in the light of this earlier analysis of the state of the art demonstrates, strikingly, the advancement that has been made.

Mr. Lane's paper thus directs the attention of the Engineering Profession at large to the vast changes in the field of masonry dam design and construction during the past thirty years. Essential water storage and power developments once thought impossible because of foundation conditions formerly considered unconquerable have been successfully consummated through an arduous and painstaking solution of the many theoretical and practical factors controlling the safe utilization of the site. During the writer's span of professional practice he has seen the design and construction of masonry dams on permeable foundations become well-established engineering practice. Thirty years ago masonry dams on other than ledge foundations were purely speculative, except for insignificant heads. Of these small dams many had been built and an astonishingly large percentage had gone down the river, for reasons usually unknown. To-day, the building of safe low dams on earth foundations is relatively common, and in the field of higher heads the upward limit has not been reached.

The results presented in this paper should not be used arbitrarily as the necessary fundamentals for the design of dams on soft foundations, owing to the considerable height and importance of many of the dams accompanying modern hydraulic projects. The design of a high masonry dam on foundations other than sound ledge rock presents a problem that differs from the design of low weirs, such as comprise more than one-half the dams listed by Mr. Lane. It would have been more desirable to have segregated these dams according to height, but even this would not tell the complete story. Each high dam on permeable foundations (much more so than in the case of low-head dams), presents an individual problem that may bear little relation to the design of any other such dam. The inclusion of a multitude of small dams in the paper adds to its value as a compendium of experience, but whether the study of the design and performance of these low-head structures adds substantially to the value of a weighted-creep ratio theory, or is actually misleading, is a question.

The securing of a high factor of safety for a low dam usually requires small investment, particularly if the depth of vertical up-stream cut-off theoretically required may be small. This is generally the case, due to the relatively great length of base required for stream-bed protection below small spillway diversion weirs on soil foundation. A few additional feet in depth of sheet-piling does not proportionately increase the cost of the cut-off and is cheap insurance. Therefore, a greater study of the higher structures may well be justified, as the cost of providing an adequate factor of safety for a high dam for power or for storage on earth foundation becomes one which may actually determine the economic feasibility of the entire development. It is fallacious and misleading to apply to the higher range of dams on

[59] "Dams on Sand Foundations: Some Principles Involved in Their Design, and the Law Governing the Depth of Penetration Required for Sheet Piling," *Transactions*, Am. Soc. C. E., Vol. LXXIII (1911), p. 175.

permeable foundations arbitrary coefficients derived from observation of a large number of unimportant low dams with heads of 15 ft, or less. A more valuable measure of proper requirements for future structures of considerable height is experience in the design and construction of similar structures and observation of their subsequent behavior over a long period of years.

The primary purpose of the design for a dam on porous foundations is to produce a structure with the largest factor of safety for the smallest money investment, within reasonable limits. Wasteful use of material on the one hand must not be encouraged; on the other hand, a design must be produced which meets all requirements for an individual site, which will be safe against unforeseen events, and which will require as little maintenance as possible.

The writer is reluctant to suggest that the author's ratios, already well below those of Bligh, should be reduced still further in general practice. However, it does appear that the rather meager data concerning failures, which have controlled his safe percolation distances are neither sufficiently authentic nor sufficiently comprehensive to justify the reliance that he has placed upon them. Undoubtedly, in the case of most of these failures, the true cause has never been fully established. Unknown defects in construction and changes in operating circumstances or natural conditions may have contributed even to those failures which are specifically classed as due to under-seepage.

Table 8 is offered to prove that the creep ratios for existing low-head dams influence more important structures unfavorably from the viewpoint of economical design, and, furthermore, to demonstrate that dams of higher heads may be built safely with lower creep ratios than those of lower heads. The tabulation includes 186 dams and power plants, on soft foundations, assembled from the writer's experience and records and from those of his associate in professional practice, A. Streiff, M. Am. Soc. C. E. Full acknowledgment is made to him for his aid in the laborious compilation of these new data, as well as to H. S. Hunt and W. G. Fargo, Members, Am. Soc. C. E., for the exact data on dams designed or constructed under their supervision. Many of these structures are quite old, and, therefore, caused the writer much difficulty in assembling correct foundation records and substructure data, but the basic information is derived from contact with each project. It is difficult to place many of these projects, accurately, in a somewhat arbitrary class of foundation material, but the grouping used in Table 8 is substantially along the same line as that of the author, and, in general, follows his classifications.

The large number of structures covered by Table 8 will justify a further examination of the safe weighted-creep ratios laid down by the author. For the purpose of making the data comparable with those of Mr. Lane, in preparing Table 8 the writer has shown, in one column, the author's recommendation for the weight of horizontal creep as one-third, although with reservations as discussed elsewhere. He also shows the actual (unweighted) creep ratio.

TABLE 8.—Comparative Data, Weighted-Creep Relations

Dam No.	Project	Stream	State	Head, in feet	Creep Distance, in Feet		Creep Ratios	
					Vertical	Horizontal	Actual	Weighted
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	(a) Dams on Very Fine Sand or Silt							
1	Oklahoma City By-Pass	North Canadian	Oklahoma	6	38	34	12.00	8.22
2	Oklahoma City Diversion	North Canadian	Oklahoma	10	40	30	7.00	5.00
3	Middletown	Miami	Ohio	10	28	39	6.70	4.10
4	San Benito Lock	Rio Grande	Texas	10	20	95.5	11.55	5.20
5	Wiggin	Bijou Creek	Colorado	11	48	41	8.09	5.55
6	Hightstown No. 1	Rock Brook	New Jersey	12	30	28	4.83	3.28
7	Otsego	Kalamazoo	Michigan	14	81	28.5	7.83	6.45
8	Barre	Prince	Massachusetts	15	50	67	7.80	4.82
	Mean for dams, 15 ft., or less	....	....	....	....	....	7.93	5.18
9	Sommerset	Apple	Wisconsin	20	40	45	4.25	2.75
10	Cotulla	Nueces	Texas	22	60	75	6.14	3.86
11	Millikin	Nueces	Texas	22	52	82	6.09	3.60
12	Anadarko	Washita	Oklahoma	28	65	57	4.36	3.00
13	Akron	Cuyahoga	Ohio	35	115	100	6.14	4.24
14	Rogers	Muskegon	Michigan	36	100	89	5.25	3.60
15	Croton	Muskegon	Michigan	39	180	275	11.67	6.95
16	Pittsfield No. 1	Ashley Brook	Massachusetts	41.5	....	....	....	....
	Mean for dams higher than 15 ft.	....	....	....	....	....	6.61	4.22
	Mean for Table 8 (a)	....	....	....	....	....	7.01	4.51
	Creep ratios recommended by Lane	....	....	....	....	....	....	8.50
	(b) Dams on Medium Sand							
17	Lake Erie	Lake Erie Inlet	Ohio	6	10	17	4.50	2.61
18	Oliver	Okanagan	British Columbia	8	15	30	5.62	3.13
19	Portland	Looking Glass	Michigan	10	36	38	7.40	4.87
20	Wichita	Wichita	Texas	12	32	180	17.67	7.67
21	Thompsonville	Bear Creek	Michigan	13	44	80	9.53	5.42
22	Tecumseh	Rasin	Michigan	15	53	56	7.25	4.75
	Mean for dams, 15 ft., or less	....	....	....	....	....	9.24	5.05
23	Pasadena	Reservoir	California	16	....	....	....	....
24	Grand Junction	Gunnison	Colorado	18	45	60	5.83	3.61
25	Berrien Springs	St. Joseph	Michigan	23	106	62	7.32	5.50
26	Rice's Rip	Messalonskee	Maine	28	96	66	5.79	4.22
27	Mio	Au Sable	Michigan	29	112	100	7.31	5.02
28	Traverse City	Boardman	Michigan	29	61	54	3.95	2.72
29	Bassano	Bow	Alberta	50	25	165	3.80	1.60
	Mean for dams higher than 15 ft.	....	....	....	....	....	5.38	3.47
	Mean for Table 8 (b)	....	....	....	....	....	6.40	3.89
	Creep ratios recommended by Lane	....	....	....	....	....	....	6.00
	(c) Dams on Coarse Sand							
30	Bound Brook No. 1	Raritan	New Jersey	6	6	24	5.00	2.34
31	Cranford	Rahway	New Jersey	6	16	20	6.00	3.78
32	Bound Brook No. 2	Millstone	New Jersey	7	8	30	5.42	2.57
33	Upper Nile	Upper Nile Canal	Colorado	10	60	31	9.10	7.03
34	Watertown	Rock	Wisconsin	15	48	20	4.53	3.64
35	Highland Park	Reservoir	Michigan	15	4	140	9.60	3.34
36	Leavittsburg	Mahoning	Ohio	15	42	58	6.63	4.08
	Mean for dams, 15 ft., or less	....	....	....	....	....	6.85	3.94
37	Caro	Cass	Michigan	17	50	22	4.53	3.37
38	Columbia	Paulins Kill	New Jersey	20	64	2	3.30	3.27
39	Gloucester	Ipswich	Massachusetts	21	42	70	5.33	3.12
40	Wausau	Wisconsin	Wisconsin	32	100	98	6.19	4.14
41	Pitt	Pitt River	California	58	136	58	3.34	2.69
	Mean for dams higher than 15 ft.	....	....	....	....	....	4.34	3.22
	Mean for Table 8 (c)	....	....	....	....	....	5.17	3.45
	Creep ratios recommended by Lane	....	....	....	....	....	....	5.00

TABLE 8.—(*Continued*)

Dam No. (1)	Project (2)	Stream (3)	State (4)	Head, in feet (5)	Creep Distance, in Feet: Vertical (6)	Creep Distance, in Feet: Horizontal (7)	Creep Ratios: Actual (8)	Creep Ratios: Weighted (9)
	(*d*) Dams on Medium Gravel							
42	Charlotte	Battle Creek	Michigan	5	8	15	4.60	2.60
43	Dimondale	Grand	Michigan	8	16	32	6.00	3.32
44	Hayden	Raisin	Michigan	8	14	25	4.88	2.78
45	Centerville	St. Joseph	Michigan	8	12	36	6.00	3.00
46	Kalamazoo	Kalamazoo	Michigan	8	17	40	7.12	3.79
47	Smithville	Grand	Michigan	8	18	25	5.38	3.28
48	Pompton Lakes	Post Brook	New Jersey	9	48	21	7.67	6.12
49	Joliet No. 2	Desplaines	Illinois	12	40	30	5.82	4.17
50	Concord	Kalamazoo	Michigan	12	36	42	6.50	4.20
51	Woonsocket	Providence	Rhode Island	13	30	36	5.07	3.24
	Mean for dams, 15 ft., or less	....	....	....	....	....	5.95	3.73
52	Watertown	Rock	Wisconsin	17	40	48	5.18	3.41
53	Crystal City No. 2	Nueces	Texas	20	62	77	6.95	4.42
54	Cartersville	Etowah	Georgia	21	42	30	3.43	2.48
55	Woodstock	Ottauquechee	Vermont	24	40	40	3.33	2.23
56	Uncas	Shetucket	Connecticut	28	90	80	6.07	4.17
57	Williams	White	Indiana	29	....	....	....	....
58	Alameda	Alameda Creek	California	31	70	140	6.77	3.76
59	Plattsburg	West Brook	New York	35	....	....	....	....
60	Grants Pass	Rogue	Oregon	38	64	70	3.54	2.30
61	Crystal City No. 1	Nueces	Texas	50	70	120	3.80	2.20
	Mean for dams higher than 15 ft	....	....	....	....	....	4.73	2.97
	Mean for Table 8 (*d*)	....	....	....	....	....	5.07	3.18
	Creep ratio recommended by Lane	....	....	....	....	....	....	3.50
	(*e*) Dams on Gravel and Sand							
62	Albion	Kalamazoo	Michigan	8	18	16	4.25	2.92
63	Fort Humphrey	Accotink Creek	Virginia	9	40	12	5.77	4.88
64	Oneonta	Susquehanna	New York	9	25	30	6.12	3.90
65	Greenville	Flat	Michigan	10	20	50	7.00	3.66
66	Balding	Flat	Michigan	10	53	39	9.20	6.60
67	Stamford	Mill Creek	Connecticut	10	20	3	2.30	2.10
68	Smyrna	Flat	Michigan	10	34	21	5.50	4.10
69	Jonesville	St. Joseph	Michigan	11	25	52	7.00	3.85
70	Flint	Flint	Michigan	11	44	40	7.63	5.20
71	Homer	Kalamazoo	Michigan	11	22	23	4.08	2.87
72	Saranac	Grand	Michigan	12	25	26	4.25	2.81
73	River Raisin	Raisin	Michigan	12	39	28	5.58	4.02
74	Allegan	Kalamazoo	Michigan	13	70	30	7.70	6.15
	Mean for dams, 15 ft., or less	....	....	....	....	....	5.92	4.11
75	Reed City	Hershey	Michigan	16	29	25	3.38	2.33
76	Ypsilanti	Huron	Michigan	17	58	28	5.07	3.97
77	Garfield	Garfield Creek	California	18	6	80	4.77	1.80
78	Joliet No. 1	Desplaines	Illinois	20	42	26	3.40	2.54
79	Gillespie	Gila	Arizona	22	60	68	5.82	3.76
80	Twin Branch	St. Joseph	Indiana	22	60	....	2.72	2.72
81	Jim Falls	Chippewa	Wisconsin	25	62	36	3.92	2.96
82	Hyatts	Oswegatchie	New York	28	40	64	2.96	2.19
	Mean for dams, higher than 15 ft	....	....	....	....	....	4.07	2.78
	Mean for Table 8 (*e*)	....	....	....	....	....	4.89	3.37
	Creep ratio recommended by Lane	....	....	....	....	....	....	....

TABLE 8.—(*Continued*)

Dam No.	Project	Stream	State	Head, in feet	Creep Distance, in Feet: Vertical	Creep Distance, in Feet: Horizontal	Creep Ratios: Actual	Creep Ratios: Weighted
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			(*f*) Dams on Boulders with Cobbles and Gravel					
83	Newton Lower Falls	Charles	Massachusetts	7	40	20	8.57	6.67
84	Rolling Fork	Rolling Fork	Kentucky	8	30	15	5.63	4.38
85	Picton	Raritan	New Jersey	8	43	19	7.75	6.17
86	Post Brook	Post Brook	New Jersey	9	50	21	7.89	6.33
87	Brome	Brome Lake Inlet	Quebec	10	38	17	5.50	4.37
88	Dunbarton	Battenkill	New York	11	36	48	7.63	4.73
89	Piedmont	Potomac	West Virginia	12	18	14	2.67	1.89
90	Sixth Lake	Sixth Lake Inlet	New York	12	18	31	4.08	2.36
	Mean for dams, 15 ft., or less	....	....	....	....	....	5.95	4.35
91	Walkill	Front Brook	New York	17	14	30	2.59	1.41
92	Flint River	Flint	Georgia	24	30	105	5.63	2.71
93	Muskogee	Reservoir	Oklahoma	30	6	105	3.70	1.37
94	Twin City	Mississippi	Minnesota	31	74	101	5.65	3.47
95	Jonquiere	Jonquiere	Quebec	35	38	66	2.97	1.72
96	Shoshone	Big Horn	Wyoming	58	67	83	2.59	1.63
	Mean for dams higher than 15 ft	....	....	....	....	....	3.69	2.01
	Mean for Table 8 (*f*)	....	....	....	....	....	4.32	2.67
	Creep ratio recommended by Lane	....	....	....	....	....	....	2.50
			(*g*) Dams on Boulders, Gravel, and Sand					
97	St. Marys	Sun	Montana	7	20	38	8.29	4.66
98	Chester	Coleman Brook	New Jersey	10	30	28	5.80	3.93
99	Shiatown	Shiawassee	Michigan	10	46	38	8.40	5.87
100	Ionia	Grand	Michigan	10	68	45	11.30	8.30
101	Mishawaka	St. Joseph	Indiana	10	34	40	7.40	4.73
102	Corbett	Shoshone	Wyoming	12	....	....	....	....
103	Lowell	Grand	Michigan	12	14	30	3.67	2.00
104	Monarch	Kalamazoo	Michigan	13	51	58	8.38	5.40
105	Plainwell	Kalamazoo	Michigan	14	32	22	3.86	2.81
106	Buchanan	St. Joseph	Michigan	14	85	24	7.78	6.50
107	Ceresco	Kalamazoo	Michigan	15	39	33	4.80	3.34
	Mean for dams, 15 ft., or less	....	....	....	....	....	6.74	4.68
108	Carson City	Fish	Michigan	20	45	61	5.30	3.27
109	Bethlehem	Ammonoosuc	New Hampshire	23	100	56	6.78	5.15
110	Bakerton	West Brook	Pennsylvania	24	....	....	....	....
111	Decorah	Iowa	Iowa	24	72	38	4.58	3.53
112	Dobbin	Stony	West Virginia	36	90	126	6.00	3.67
113	Lansford	Mesquahoning Cr.	Pennsylvania	50	....	....	....	....
	Mean for dams higher than 15 ft	....	....	....	....	....	5.71	3.89
	Mean for Table 8 (*g*)	....	....	....	....	....	6.26	4.30
	Creep ratio recommended by Lane	....	....	....	....	....	....	....
			(*h*) Dams on Medium Clay					
114	Somerville	Raritan	New Jersey	8	27	13	5.00	3.92
115	Hightstown No. 2	Rock Brook	New Jersey	10	30	20	5.00	3.67
116	Cedar Rapids	Cedar	Iowa	10	36	67	10.30	5.83
117	South Bend	St. Joseph	Indiana	10	37	45	8.20	5.20
118	Elk Rapids	Elk Lake Outlet	Michigan	11	32	49	7.36	4.40
119	Springfield	Black	Vermont	12	28	23	4.25	2.97
120	Three Rivers Power House	St. Joseph	Michigan	12	53	40	7.75	5.52
121	Utah Avenue	Flint	Michigan	12	106	85	15.90	11.20
122	Redland	Gunnison	Colorado	13	50	57	8.23	5.31
	Mean for dams, 15 ft., or less	....	....	....	....	....	8.15	5.43

TABLE 8.—(*Continued*)

Dam No.	Project	Stream	State	Head, in feet	Creep Distance, in Feet		Creep Ratios	
					Vertical	Horizontal	Actual	Weighted
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
			(*h*) Dams on Medium Clay — (*Continued*)					
123	Goshen	Elkhart	Indiana	16	40	69	6.82	3.94
124	Scranton	Roaring Brook	Pennsylvania	17	31	12	2.53	2.06
125	Elkhart	St. Joseph	Indiana	18	32	30	3.44	2.34
126	Gonzalez No. 1	Guadalupe	Texas	18	52	53	5.83	3.87
127	Swift Current	Swift Current	Saskatchewan	20	68	68	6.80	4.53
128	Trowbridge	Kalamazoo	Michigan	21	42	32	3.52	2.51
129	Stronach	Manistee	Michigan	23	37	30	2.91	2.04
130	St. Paul	Reservoir	Minnesota	25	10	120	5.20	2.00
131	Webber Spillway	Grand	Michigan	26	60	52	4.31	2.97
132	Webber Power House	Grand	Michigan	26	33	43	2.92	1.82
133	Texas Power No. 5, Spillway	Guadalupe	Texas	27	73	63	5.04	3.48
134	Texas Power No. 5, Power House	Guadalupe	Texas	27	49	49	3.63	2.42
135	Massillon	Nimisilla Creek	Ohio	28	43	59	3.64	2.24
136	Kearsley	Kearsley Creek	Michigan	28	138	106	8.72	6.18
137	Hunt Devel. No. 5, Power House	Guadalupe	Texas	28	49	57	3.79	2.43
138	Hunt Devel. No. 4, Spillway	Guadalupe	Texas	28	57	68	4.46	2.85
139	Hunt Devel. No. 4, Power House	Guadalupe	Texas	28	49	57	3.79	2.43
140	Hunt Devel. No. 5, Spillway	Guadalupe	Texas	28	57	68	4.46	2.85
141	Cleveland	Hinckley	Ohio	29	24	32	1.93	1.20
142	Texas Power No. 4, Spillway	Guadalupe	Texas	30	36	62	3.27	1.89
143	Texas Power No. 4, Power House	Guadalupe	Texas	30	62	61	4.10	2.74
144	Cuero	Guadalupe	Texas	30	40	35	2.50	1.72
145	McQueeney, Power House	Guadalupe	Texas	31	43	57	3.23	2.00
146	McQueeney, Spillway	Guadalupe	Texas	31	60	70	4.20	2.69
147	Prince Albert	N. Saskatchewan	Saskatchewan	31	50	125	5.65	2.96
148	Grays	Black Creek	New York	32	40	112	4.75	2.42
149	Loud	Au Sable	Michigan	32	50	250	9.37	4.17
150	Burnham	Sebasticook	Maine	34	68	105	5.09	3.03
151	Alcona	Au Sable	Michigan	35	180	200	10.86	7.05
152	Cooke	Au Sable	Michigan	40	76	289	9.12	4.30
153	Five Channels	Au Sable	Michigan	44	90	255	7.84	3.97
154	Williamson No. 1	Sandy Creek	Texas	45	72	45	2.60	1.93
155	Texas Power No. 1, Spillway	Guadalupe	Texas	46	55	73	2.78	1.73
156	Texas Power No. 1, Power House	Guadalupe	Texas	46	74	67	3.06	2.10
157	Foote	Au Sable	Michigan	48	150	200	7.29	4.51
158	Junction	Manistee	Michigan	50	84	48	2.64	2.00
159	Mathis Dike	Tallulah	Georgia	50	199	165	7.28	5.10
160	Williamson No. 2	Sandy Creek	Texas	64	50	84	2.10	1.22
161	Hodenpyl	Manistee	Michigan	65	144	381	8.08	4.20
162	Sand Springs	Shell Creek	Oklahoma	65	....	....	....	....
163	Rodriguez	Tijuana	Mexico	250	400	275	2.70	1.97
	Mean for dams higher than 15 ft.	....	....	....	....	....	4.58	2.83
	Mean for Table 8 (*h*)	....	....	....	....	....	4.80	2.98
	Creep ratio recommended by Lane	....	....	....	....	....	....	2.00
			(*i*) Dams on Very Hard Clay and Hardpan					
164	Patillas	Patillas	Puerto Rico	5	16	25	8.20	4.87
165	Key West	Ocean Inlet	Florida	8	6	10	2.00	1.17
166	Batavia	Towanda Creek	New York	11	16	35	4.64	2.52
167	Fullerton	Cedar	Nebraska	13	24	34	4.46	2.72
168	Baltic No. 1	Shetucket	Connecticut	15	16	20	2.40	1.51
169	Wakeman	Vermilion	Ohio	15	30	42	4.80	2.94
	Mean for dams, 15 ft., or less	....	....	....	....	....	4.10	2.44

TABLE 8.—(*Continued*)

Dam No.	Project	Stream	State	Head, in feet	Creep Distance, in Feet		Creep Ratios	
					Vertical	Horizontal	Actual	Weighted
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	(*i*) Dams on Very Hard Clay and Hardpan — (*Continued*)							
170	Newport	Sugar	New Hampshire	18	42	56	5.45	3.37
171	Wonder Lake	Nippersink Cr.	Illinois	21	25	68	4.43	2.27
172	New London	Bogue Brook	Connecticut	22	26	30	2.55	1.64
173	Grand Ledge	Lookingglass	Michigan	23	16	30	2.00	1.13
174	Amherst	Amherst Creek	Nova Scotia	25	40	50	3.60	2.26
175	Vandalia	Milk	Montana	27	113	191	11.26	6.65
176	Juniata	Juniata	Pennsylvania	28	....	....	....	....
177	Baltic No. 2	Shetucket	Connecticut	30	28	40	2.27	1.38
178	Pittsfield No. 2	Mill Brook	Massachusetts	30	30	38	2.27	1.42
179	Logan	Logan	Utah	31	26	56	2.65	1.44
180	Mayfield	Mayfield	New York	32	50	105	4.85	2.66
181	River Falls	Conecuh	Alabama	37	20	100	3.24	1.44
182	St. Francis Dam	St. Francis	Quebec	40	40	48	2.20	1.40
183	Prattville	Tallapoosa	Alabama	42	80	110	4.52	2.78
184	Girard	Meander Creek	Ohio	60	20	68	1.47	0.71
185	Rapidan	Blue Earth	Minnesota	64	....	....	....	....
186	La Prele	La Prele	Wyoming	135	....	....	....	....
	Mean for dams higher than 15 ft	....	....	....	....	....	3.53	2.02
	Mean for Table 8 (*i*)	....	....	....	....	....	3.60	2.08
	Creep ratio recommended by Lane	....	....	....	....	....	....	1.60

The writer wishes to make it very plain that in giving the mean creep ratio, whether weighted or actual, it should not be inferred that any such rule-of-thumb determination of a safe ratio is even implied for application to other projects on seemingly similar foundation materials. It is well to state at this point that the great variation in the creep ratios for individual structures on similar foundation material has been due to many factors, not the least of which has been the profitable knowledge of the methods of design and construction successfully followed in earlier structures. Furthermore, particularly in the case of the weighted-creep ratios, the shape of the structure, the details of the design, the necessary erosion protection, and numerous other circumstances have been material factors in these variations.

A few of the dams tabulated have not been included in the computations of the ratios; foundation data were not dependable, and, in some cases, the cut-off wall was carried into impervious material. As in Mr. Lane's lists, they are included without analysis. In a few of the structures the use of weep-holes in the floor tends to shorten the creep distance somewhat, but this has been ignored in the analysis, as was also done by Mr. Lane in his tables.

Table 8 shows in a striking manner that a creep ratio for all dams on any classification of foundation material, if based on a mean of the existing dams, is greatly increased by the inclusion of the structures with heads of 15 ft, or less. Although the author did not determine his recommended new weighted-creep ratios from the mean creep ratios of his Table 1(*a*), 1(*b*), 1(*c*), and 1(*d*), (being governed instead by the ratio at which no dam failure has been

recorded) the relatively few failures which have controlled his ratios, in most cases, cannot be conclusively shown to have occurred from piping alone. A creep ratio might well be determined from a consideration of the behavior of successful dams, if they are included in a large enough foundation group and are old enough to justify confidence.

For example, Table 1(*d*), from which data the author's creep ratios for the three classifications of "very fine sand or silt," "fine sand," and "medium sand," are derived, contains data on sixty-five dams; nine failed, most of them being low-head Indian weirs. Of the fifty-six remaining, thirty-nine, or 70%, have heads of 15 ft or less, leaving only seventeen relatively important non-failures in Mr. Lane's three highest-ratio foundation classifications. In Table 1(*c*) (coarse sand), 59% of the non-failures are 15 ft, or less, in height; in Table 1(*b*) (gravel, cobbles, and boulders), 69%; and, in Table 1(*a*) (clay and hardpan), 38 per cent. Of the four tables combined, the low dams account for 56% of the numerical total. Although insisting on conservatism in the design of a dam of any size on permeable foundations, the writer believes that creep ratios so heavily influenced by a high proportion of low-head structures may penalize, unnecessarily, the designers of higher-head developments.

TABLE 9.—COMPARISON OF CREEP RATIOS

Table No.	Foundation classifications	Mean Creep Ratios: Dams of 15 Feet or Less: Number of dams	Weighted creep	Actual creep	Dams Higher Than 15 Feet: Number of dams	Weighted creep	Actual creep	All Heads Combined: Number of dams	Weighted creep	Actual creep	Proposed Ratios: Lane	Bligh	Griffith
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
8(*a*)	Very fine sand and silt	8	5.18	7.93	7	4.22	6.61	15	4.51	7.01	8.50	18	11.6 to 12.8
	Fine sand	0	....	....	0	....	....	0	....	....	7.00	15	10.0 to 11.2
8(*b*)	Medium sand	6	5.05	9.24	6	3.47	5.38	12	3.89	6.40	6.00	.....	.....
.....	Writer's group, Table 8(*a*) and 8(*b*)	14	5.15	....	13	3.88	....	27	4.22	....	....	.....	.....
.....	Author's group, Table 1(*d*)*	41	9.90	....	10	6.88	....	51	8.95	....	....	.....	.....
.....	Both groups combined	55	8.66	....	23	4.92	....	78	6.80	....	....	.....	.....
8(*c*)	Coarse sand	7	3.94	6.85	5	3.22	4.34	12	3.45	5.17	5.00	12	8 to 9.6
....	Author's group, Table 1(*c*)*	7	8.72	....	5	4.80	....	12	6.14	....	....	.....	.....
....	Both groups combined	14	6.20	....	10	4.00	....	24	4.72	....	....	.....	.....
....	Fine gravel	0	....	....	0	....	....	0	....	....	4.00	.....	.....
8(*d*)	Medium gravel	10	3.73	5.95	8	2.97	4.73	18	3.18	5.07	3.50	.....	.....
8(*e*)	Gravel and sand	13	4.11	5.92	8	2.78	4.07	21	3.37	4.89	....	9	6.4
....	Coarse gravel, including cobbles	0	....	....	0	....	....	0	....	....	3.00	.....	.....
8(*f*)	Boulders, cobbles, and gravel	8	4.35	5.95	6	2.01	3.69	14	2.67	4.32	2.50	.....	4.8
8(*g*)	Boulders, gravel, and sand	10	4.68	6.74	4	3.89	5.71	14	4.30	6.26	....	4 to 6	6.4
....	Writer's group, Table 8(*d*), 8(*c*), 8(*f*), and 8(*g*)	41	4.22	....	26	2.80	....	67	3.33	....	....	.....	.....
....	Author's group, Table 1(*b*)*	44	7.14	....	11	3.12	....	55	5.13	....	....	.....	.....
....	Both groups combined	85	5.62	....	37	2.92	....	122	4.05	....	....	.....	.....
....	Soft clay	0	....	....	0	....	....	0	....	....	3.00	.....	.....
8(*h*)	Medium clay	9	5.43	8.15	40	2.83	4.58	49	2.98	4.80	2.00	.....	.....
	Hard clay	0	....	....	0	....	....	0	....	....	1.80	.....	.....
8(*i*)	Very hard clay and hardpan	6	2.44	4.10	14	2.02	3.53	20	2.08	3.60	1.60	.....	.....
....	Writer's group, Table 8(*h*) and 8(*i*)	15	4.20	....	54	2.64	....	69	2.76	....	....	.....	.....
....	Author's group, Table 1(*a*)*	22	5.45	....	21	2.30	....	43	3.18	....	....	.....	.....
....	Both groups combined	37	4.95	....	75	2.57	....	112	2.88	....	....	.....	.....

* Omitting writer's data.

In the mean ratios derived from Mr. Lane's tables this influence of the low-head dams is even more pronounced than in the case of Table 8, as shown by Table 9. For convenience of reference and for the purpose of comparison the writer's data have been grouped in the same four general foundation classifications as the author's tables. It should be noted in this connection that Table 8 includes data concerning approximately ninety dams which were originally furnished to Mr. Lane and included in the complete report on file at Engineering Societies Library. The remainder are new to the discussion as foundation details were unavailable in time to be included by the author. To prove the writer's view that, in accordance with his own experience, high dams may be designed properly with lower creep ratios, whether weighted or actual, than low dams, he has segregated into "Author's Group" (Table 9) the mean ratios for those dams in which data did not originate with the writer and Mr. Streiff, and has shown in "Writer's Group" the mean ratios of all those dams covered by Table 8. All ratios are for successful structures.

To obtain the greatest possible accuracy and scope for Table 9, of the 186 listed structures in Table 8, eleven have not been included because of uncertainty as to exact percolation factors; and six shown are power houses with creep ratios differing from those of the dams built in conjunction with them. In computing the means for the author's group (being the basic 140 dams remaining in his original Tables 1(*a*), 1(*b*), 1(*c*), and 1(*d*), after those on the writer's lists have been eliminated) twenty-one creep ratios have been added which occurred in different parts of certain individual dams. By the foregoing method 336 different creep ratios are analyzed in Table 9. Therefore, in support of the writer's conclusion that high dams can be constructed safely with lower creep ratios than low dams, he not only relies upon his own lists (Tables 8 and 9) (11 dams with indeterminate ratios have not been analyzed) showing 77 structures 15 ft or less in height and 98 (or 56%) with heads greater than 15 ft; but also upon the author's group of 161 creep ratios, of which 114 are from dams of 15 ft, or less, head, and 47 (or 29%) with heads greater than 15 ft.

The greater disparity between the mean creep ratios of the low-head dams and the dams of higher head in Mr. Lane's tables seems partly due to the inclusion in his lists of a large number of small Indian and Egyptian weirs and low U. S. Government dams, having quite high ratios. Many of these structures are on pile foundations or consist largely of considerable horizontal hearth or paving, having horizontal percolation distances that are probably of slight value; yet they heavily influence the mean ratios of his tables. Furthermore, the high proportion of 71% of low dams to 29% of high ones widens the gap. As hereinafter shown, a favorable influence for the higher-head structures is their increasing value of horizontal creep due to properly designed supporting floor construction.

From the multitude of low weirs that have failed in the past much valuable information might have been secured. However, their small size, remoteness, and relative unimportance permitted the event to pass usually with only local notice. In most cases the true cause of the failure was never determined

and recorded. Valuable lessons may be drawn from a careful study of the failure of the most obscure dam on permeable foundations, and it is regrettable that no representative of the Engineering Profession at large has had the duty of analyzing such minor failures. For many years the writer has assembled data on reported dam failures which now cover about 300 instances. In less than 15% of the cases was the cause of failure ever determined and published in such a form as to be useful or even available to the Engineering Profession. Only twenty-one failures on porous foundations are listed by the author, and the writer ventures the statement that the direct cause of few of these failures can be attributed, exclusively and with certainty, to excessive under-seepage.

The difficulty that would be experienced in applying Mr. Lane's weighted-creep ratios from Table 3 to actual foundations will be the same as the writer has had, for example, in classifying his sand structures according to the four sand classes cited by the author. This is not mentioned in a critical sense, as all engineers experienced with foundation materials, particularly complex river-bed deposits, realize the impossibility in most cases of placing any specific site entirely under any one of the fourteen classifications used by Mr. Lane. There seems to be no exact method of classifying different materials with accuracy or uniformity for the determination of creep resistance. Two sands in different localities, of the same fineness or effective size, may have entirely different degrees of percolation resistance due to the natural bedding and stratification, or to the different shapes and irregularities of the grains. Divergence of views as to the proper foundation classification into which to cast any single structure built on varying material is naturally to be expected. The writer doubtless has placed certain of his structures which also appear in the author's tables in a different classification of foundation material than did Mr. Lane for establishing the data in his Table 3, but the foundation classifications for the dams incorporated in the writer's tables were based upon an examination of the foundation material in the field in nearly all cases. However, this has little effect on the divergence of the mean ratios for low and high dams.

Except in certain localities, such as parts of Michigan, for example, and in some of the Southwestern States, river-bed deposits are generally too heterogeneous to permit the safe application of any arbitrary creep ratio unless the requisite creep distance for the most porous deposits to be expected is applied. Mr. Lane properly offers his weighted-creep values for use only where accurate foundation data as to the extent and position of underlying impervious strata are not available, or where the data do not indicate whether or not such strata exist. The extreme variation of foundation material that is so frequently encountered at any one site serves to emphasize the writer's assertion that each dam on difficult foundations presents an individual problem. No solution of any individual problem can be for general application. It is applicable only to identical foundation material and identical design of the dam itself, including height, length of base, spillway requirements, back-water conditions, etc. The writer doubts that identical conditions will ever be found for two structures.

The proper classification of foundation types has probably been one of Mr. Lane's most difficult tasks. To force a listed dam somewhat arbitrarily into any classification might well result in indicating a structure with an impaired factor of safety against percolation, or, conversely, as having been "saddled" with an excessive construction cost, as the case might be.

At this point the writer desires to emphasize that nothing can take the place of careful and complete soil investigations as a prerequisite for the design of any dam on porous or yielding foundations. Based upon the writer's examination of many dam foundations at sites that were developed eventually, he estimates that not more than 15% of these projects had a foundation material that, during the actual construction of the work, was found to be uniformly of the anticipated characteristics as indicated by previous exploration. Of course, where such unexpected diversity of material is found the worst condition must be met by subsequent revisions of the design. Because of the probability that even the most careful sub-surface investigation will fail to reveal lenticular deposits or strata of a more porous material than that indicated by the exploration, caution must be exercised in selecting a creep ratio. Greater conservatism must be used in the design of permeable foundation dams than in the case of almost any other engineering structure.

Mr. Lane understates the important effect which the design of the superstructure may have on the value of horizontal creep. Probably this is one reason why most of the dams in Table 8 show creep ratios so much lower in general than those recommended by the author, when his weighted value of one-third is applied to the horizontal percolation-resisting elements of these dams. Experience has convinced the writer that proper design and construction are much more important than theoretical considerations relating to creep ratios, although the latter should not be neglected as an important, although not necessarily the controlling, factor in the final design.

As stated by the author, the best shape of dam must first be determined to meet other conditions, before the depths of the vertical cut-offs are established. The escape velocity of water beneath the dam is controllable by the actual design of the superstructure as well as by the design and location of the cut-off walls. Moreover, the depth to which the up-stream cut-off wall must be carried (and the down-stream one also, if one is used), is controlled by the shape of the superstructure of the dam itself and its resulting length of base. This design, in turn, is governed by factors which will vary widely for different sites, thus nullifying any arbitrary ratio based on consideration of the foundation material alone. These factors include the quantity of flood water to be discharged and the resulting shape of the rollway; the details of stilling pools, or other energy-dissipating devices; back-water conditions; the geographical location of the dam as to ice-pressure conditions; the type of crest gates, piers, or bridge that are to be superimposed on the structure; and the bearing value of the foundation itself. It is also necessary to determine the permissible limits of leakage under the structure, which, depending somewhat upon the economic factors, may be extended very close to the safety point. These are among the elements which must first be determined before

a solution of the cut-off problem is undertaken, and the results of these investigations will usually go far toward determining the proper value to be given to the resulting horizontal creep.

This point is made clear by an example, which is not the most extreme that could be selected. Consider the design of the Cooke Dam on medium clay (Dam No. 152, Table 8). The exigencies of the site required that the spillway be designed to care for a $12\frac{1}{2}$-ft depth of flood discharge. Then, scour prevention forced such an added base length as to create a horizontal creep distance of 289 ft for a 40-ft head, which alone gives the structure a creep ratio of 7.25, actual, and 2.41, weighted. The addition of the necessary minimum of vertical cut-off walls permitted by good judgment increased the creep ratio to 9.12, actual, and 4.30, weighted. Contrast this with the weighted mean of 2.83 for forty high dams on similar material in Table 8, and with the author's recommendation of 2.00, and the basis becomes evident for the writer's contention that design controls the creep ratio to a far greater extent than creep ratios control the design.

A further example will suffice. On medium sand, the 25-yr old Bassano Dam (Dam No. 29, Table 8) has a weighted-creep ratio of 1.60, 42% of the actual creep ratio of 3.80. On the other hand, the Berrian Springs Dam (Dam No. 25) has a weighted-creep ratio of 5.50, 75% of the actual creep ratio of 7.32. Both are considered equally good designs. Each misses the mean weighted creep of 3.47 for their class by a wide margin, and neither reaches the author's recommendation of 6.00.

Thus, it becomes also readily apparent that empirical creep ratios in hands unskilled in the manifold requirements for correct designs for erodible and permeable foundations may become a real danger. It is the solving of the various problems of design for each individual site that causes the great variation in creep ratios within any one foundation class in Table 8. Theoretical considerations and laboratory conclusions are valueless under these practical circumstances.

Nearly all the writer's 186 listed dams have certain design characteristics in common. Substantial vertical cut-offs are located at the up-stream and down-stream ends of continuous, well reinforced, foundation slabs. These foundation slabs are supported directly by the foundation material and bear heavily upon it, and are often keyed to it by intermediate shallow walls. Although deep-river bed deposits are usually of a rather heterogeneous nature, the method of their deposition and the long period of time during which settlement by saturation has been occurring, have given to most sites a well equalized bearing value throughout their area despite the varying resistances to percolation. Although a value of one-third for percolation distance along the under side of a concrete hearth or apron may be justified, the writer is not willing to accept so low a value where this horizontal surface is the actual foundation slab of a dam of moderate height. To carry the author's theory of a value of one-third for horizontal creep a step further, the value of vertical creep should be increased according to the depth to which the cut-off wall is carried. It would be quite logical, and fundamentally consistent with

Mr. Lane's theory, to give the percolation path at the bottom of the deepest cut-off wall a value of unity, and then to decrease this value progressively for the upper portions of the cut-off to its juncture with the dam, and continue some progressive reduction in the creep value to the zero point where the percolation emerges freely beyond the down-stream toe. The writer agrees with the author, however, that percolation resistance along the line of creep is probably less than through the foundation material itself, which path is in most cases that of least resistance. This condition is quite variable, however, depending almost entirely on the texture of the foundation material.

For a continuous-floor dam, properly designed and constructed, the horizontal creep value should be more than one-third compared with the vertical creep, whereas in the case of any type of dam having a base supported on piles horizontal creep has little or no value. This is true, of course, in the case of structures with supporting buttresses that are carried on piles, with no connecting slab, and in those rare cases where the buttresses rest upon spread footings. Where the writer refers to a floor, it is intended to mean only the characteristic continuous, monolithic, reinforced concrete foundation slab upon which the typical Ambursen type of dam for soft foundations is constructed. He does not refer to what the author terms "the wide masonry-floor type" which is understood to describe the so-called Indian weir structures with a long down-stream hearth or paving serving as a stilling pool. The writer, like Mr. Lane, places little reliance on the value of horizontal creep beneath such hearths.

The continuous floor of an Ambursen dam, whereby the weight of the structure and its water load is transmitted directly and entirely to the underlying foundation material without the use of bearing piles, has also a direct and favorable bearing on the usefulness of the vertical creep. In the case of many, if not all, permeable soil foundations there is an inevitable displacement of foundation materials or a transfer of foundation pressures caused by the application of the weight of the dam and its water load as a surcharge. This displacement, consolidation, or flow of foundation material tends to tighten the contact against both the up-stream and the down-stream faces of a (somewhat flexible) sheet-pile, up-stream, cut-off wall, and against at least the down-stream side of a masonry cut-off wall. This transmission of pressure through foundation material is so thoroughly established, it seems obvious that the application of the weight of the dam and its load to a soil foundation must inevitably retard the rate of seepage against all vertical faces of cut-off piling, whether at the heel or at the toe of the dam. Accepting this as a certainty, there must also be some consolidation at least within the upper portion of the foundation material itself to which the load of the dam is applied, with resulting decrease in voids and a tighter contact against the foundation slab, thus giving the horizontal creep a high value. The drop in upward pressure below some horizontal concrete surfaces, as stated by Mr. Lane, may be almost zero in certain cases, but where this concrete surface forms the properly constructed supporting base of the dam itself, and the structure is of substantial height, different results are obtained.

The author's quoted statement of Mr. Griffith, that "the line of creep is subject to a greater pressure where carried to a greater depth" comes from the same logical reasoning.

As shown in Fig. 10, by careful designing and the use of a continuous floor-slab to support the buttresses (a design which possesses many structural

FIG. 10.—TYPICAL AMBURSEN FLOOR-SLAB UNDER DAM 135 FEET HIGH, LA PRELE, WYOMING; CLAY AND HARDPAN FOUNDATIONS; BUILT IN 1908.

and construction advantages as well as usually making for economy as compared to a piled foundation), the writer has constructed dams successfully on saturated fine sand or silt, that have stood for more than twenty years.

The writer's preference for the use of a continuous foundation slab of reinforced concrete is due not only to his belief that this form of structure distributes the load better over the undisturbed foundation material than a piled foundation (thereby increasing its resistance to percolation instead of decreasing it), but that it further affords an admirable tie for the entire structure, acting somewhat as the bottom chord of a truss. Its value in the case of a blowout caused by piping was fully demonstrated in 1908 at Pittsfield, Mass., where the undamaged structure spanned a gap in the foundation material 50 ft in width, after the reservoir emptied safely. Repairs to the cut-off and replacement of supporting foundations were made, and this dam has been in successful use ever since.

Griffith's valuable experiments correctly pointed out that by loading the foundation material the tendency to blow out was decreased. The resis-

tance to flow is increased materially, therefore, if the weight of the dam and the water load are transmitted directly through the floor-slab to the foundation. The use of piles is to be avoided owing to the inevitable tendency of this type of construction to cause roofing. If the foundation material is properly confined by a suitable arrangement of cut-off walls, bearing piles can serve no useful purpose. As Professor W. M. Patton pithily wrote, "Sand will hold your structure if you can hold the sand." Of the 186 earth-foundation dams listed herein only eight are on piles, and of late years the writer has abandoned their use entirely in designing the conventional Ambursen type of dam. Where used, the value of horizontal creep under the dam is negligible. While the use of bearing piles beneath a gravity dam is usually required because of the heavy and unequally distributed dead load of the structure itself, no such design (except of relatively low height) can be constructed safely on a soil foundation without excessive cost. Further, just as the driving of timber or of steel sheet-piling for a cut-off wall may impair its own value as a medium for increasing percolation distance by providing a smooth surface along which water under pressure may rise from one stratum to another, so may the driving of bearing piles under a dam likewise afford a shorter upward path for percolation along their sides. These perforations through varying strata may serve as pipes to the roofed space under the piled dam.

As to the use of spread buttress footings beneath a dam for soil foundations, without piles, the writer has always avoided this dangerous form of construction. Little, if any, initial economy is obtained and, without "splitting hairs" as to the exact weight to be given to horizontal creep beneath a foundation floor, the latter has numerous advantages and no disadvantages, when compared to spread footings.

The writer agrees with Mr. Lane that a single deep up-stream cut-off wall has greater value for percolation resistance than the same total vertical depth divided into two or more walls, not only because it avoids the possibility of short-path travel, but because the resistance to percolation increases in a given material at a greater depth. The connection of the dam superstructure proper to the top of the cut-off wall should be made so as to permit slight movement without rupture of the joint and loss of water-tightness. There is nearly always some measurable although slight subsidence of the foundation material as the weight of the dam with its water load is applied. The author's statement that, "in order to use these values with safety the cut-offs must be of solid masonry built in contact with the sides of the trench, * * *," emphasizes the importance of proper design and construction methods for this critical part of the dam. Steel sheet-piling has almost entirely displaced Wakefield or other timber sheet-piling for cut-off walls, not only because of the better sections and interlocks that are marketed to-day, but because of the extremely bad record of timber sheet-piling when used in connection with hydraulic structures. Nevertheless, steel sheet-piling still leaves much to be desired as a cut-off material, even under the most favorable driving conditions.

The writer considers the up-stream cut-off the most valuable for restricting, to a permissible quantity, the seepage through a permeable foundation. Nevertheless, he has always considered it good practice in the case of a floor dam also to install substantial sheet-piling (or a masonry cut-off) either beneath the down-stream edge of the foundation slab itself, or at the down-stream edge of the stilling pool, or hearth, and in many cases at both points. This has usually been done not merely to increase the effective creep distance, but to protect the structure against scour. Mr. Lane's statement that his creep-ratio values are based on the assumption of "competent supervision during construction and efficient maintenance afterward," affords the writer the opportunity to state that, improbable as it might seem to the layman or to an engineer not experienced in this field, these two factors are not the easiest to obtain for dams on earth foundations. Obviously many, if not most, of the failures incorporated in Mr. Lane's tables, were due to lack of one or the other. Over a long period of time the writer has observed that adequate maintenance of dams on erodible foundation material is by no means always assured. This is true principally in the case of the smaller structures, although it is also found often on larger projects where there is a permanent back-water condition making close observation of scour inconvenient. As this erosion, of course, occurs even in rock foundations below important and adequately maintained structures (notable examples being the McCall Ferry Dam, in Pennsylvania, and the Keokuk Dam, in Iowa), a sturdy down-stream cut-off is always advisable for earth foundations.

The conclusion might be drawn from the cumulative weight of the behavior records of 186 dams on permeable foundations listed by the writer (many having been in use for more than 25 yr) that a more or less arbitrary creep ratio, weighted or otherwise, based upon an analysis of Tables 8 and 9 would be safe. However, the writer is not willing to advocate this practice yet for the reason, perhaps repetitiously mentioned, that every individual dam on soil foundations is a special and separate problem in itself that may have little or no relation to any other such structure. Therefore, until considerably more research has been done, and more is known as to the actual behavior of under-seepage beneath dams of substantial height under the many different conditions controlling its flow, the writer is not prepared to recommend any reduction for general use from the figures tentatively set up by Mr. Lane. Certainly, his new weighted-creep ratios for the more permeable foundations seem adequate, and for the higher dams his ratios might be substantially reduced in special cases if proper design, construction, and maintenance were assured. This is in disagreement with the author's views that his values are suitable for major structures, and that smaller values may be used for less important ones. The writer believes that higher factors of safety for smaller dams can usually be had so cheaply that they are justified, since any earth foundation development, in the last analysis, must be classed as hazardous.

Mr. Lane's statement that some light may be thrown on the relative value of vertical and horizontal creep through a study of upward pressure on actual

dams is correct. The most authentic and reliable information that can be procured is through the medium of tests of pressure and flow under structures, such as the system installed in the Rodriguez Dam, in Mexico (250 ft high) to permit the observation and recording of upward pressure beneath the base of this dam, as well as at the bottom of its cut-off wall, 300 ft deep. Similar recording installations where possible should be provided in all future dams 15 ft, or more, in height on permeable foundations. Nevertheless, Mr. Lane properly states that "while research along this line should be given every encouragement, the main reliance in dam design, for a long time in the future, must be on a somewhat empirical basis." Small-scale model experiments, laboratory tests, and the flow-net or electric analogy methods have been in common use for many years. They still remain merely aids to good judgment and experience which latter must form the basis for successful design and construction of masonry dams on soil foundations. In inexperienced hands, the application of such theoretical factors may be more dangerous than beneficial. If sufficiently complete information were available as to the exact conditions of the foundation material, greater reliance might be placed on the electric analogy or flow-net methods, but in the presence of such detailed knowledge, adequate and economical design for percolation resistance could be made without resort to them.

The treatment of each class of foundation material must be fundamentally a matter of sound judgment based on experience. Economy in design and construction (which, in the last analysis, is the objective of most engineering work) should flow only from this source. The writer's experience with the design, construction, and maintenance of a large number of dams on difficult foundations, has led him to conclude that no method of theoretical or empirical analysis will ever be satisfactory and uniformly applicable to such structures. Much more study of actual rates and the behavior of under-flow and of upward pressures is needed before Mr. Lane's recommendations can be accepted entirely; but as more quantitative and qualitative data become available (particularly as to failures), the writer is convinced that further reductions will be made in the percolation ratios now considered necessary. The ratios of both Bligh and Griffith have the serious weakness of being based on observations merely of a relatively few low-head dams, and their available data were also restricted as to classes of foundation materials. Moreover, the number of observable structures was insufficient to justify any attitude other than the greatest conservatism.

In conclusion, where conditions vary so widely between different sites and structures there are naturally divergencies of opinion as to what constitutes the best, or at least the favored, practice from the viewpoint of different engineers trained in this field. All may be meritorious, but they must be so proved in practice and by the greatest test of all—time. The writer, because of his nearly thirty years experience with the Ambursen type of dam, has naturally favored the type of structure and details of design and construction that in his experience have proved satisfactory. He does so with the greatest respect for the individual preferences and opinions of other engineers who may be skilled and experienced in the same field.

Among the most important conclusions and advice to be drawn from Mr. Lane's invaluable paper are the following:

(1) Explore as fully as may be economically feasible, the sub-surface area beneath and adjacent to the dam;

(2) Use a continuous well-reinforced concrete floor beneath any buttress dam for soft foundations, regardless of differences of opinion as to the exact creep value of such construction;

(3) Avoid the use of pile foundations or spread footings for dams of any but the lowest heads;

(4) Avoid reliance on horizontal creep under long down-stream pavements or stilling basins, particularly where articulated;

(5) Take advantage of the economical up-stream impervious fill to increase seepage distance;

(6) Rely principally on a properly constructed up-stream masonry cut-off wall for control of under-seepage and piping;

(7) Install a good down-stream cut-off as insurance against removal of foundation material by scour, and as Mr. Lane's "second line of defense"; and,

(8) Where rock or other impervious material may be reached without prohibitive cost, carry a substantial concrete cut-off wall down to it, regardless of theoretical creep limitations.

These principles are among the fundamentals for soft foundation dam design and construction, and are believed worthy of emphasis. There is no structure justifying a higher factor of safety than a dam on soil foundations nor, as may be judged from the diversity of views shown, one less susceptible to determination based upon laboratory or theoretical considerations. Mr. Lane's paper, serving as it does to call attention to the great advances made during thirty years in the treatment of difficult dam foundations, is not merely a tribute to the pioneering work of those engineers who had the courage to break with precedent and devise new types and methods for solving the problems inherent to this field of engineering. It also serves as an incentive to others to aid in the expansion and distribution of this knowledge by research, experimentation, and observation. Experience breeds confidence, and confidence broadens the engineering horizon. The paper is a useful encyclopædia of experience gained in developing this difficult field, in addition to the unquestionable value of Mr. Lane's effort to fix a needed new standard for security from under-seepage, with resulting economies in construction cost. Further advancement must follow as a direct result. This painstaking and laborious paper coincides with a transition period in the art of all dam design, and is presented at a time when similar research is most needed. The profession is indebted to Mr. Lane for his efforts.

ALEXANDER POTTER,[60] M. AM. SOC. C. E. (by letter).[60a]—In 1911, in a discussion[61] of the paper by Mr. Arnold C. Koenig, the writer ventured the prediction that the increasing municipal and commercial needs of rapidly

[60] Cons. Engr., New York, N. Y.

[60a] Received by the Secretary March 26, 1935.

[61] *Transactions*, Am. Soc. C. E., Vol. LXXIII (1911), p. 191.

developing areas of the country would force the harnessing of rivers and streams with unstable foundations, correct solutions for which had not then been found. That this prediction was timely has been borne out by developments during nearly twenty-five years which have elapsed since that time. It is evidenced, furthermore, by the long list of successfully constructed dams shown by the author, which tabulation adds a double value to his paper.

In compiling the mass of data presented in this paper, occasional errors are unavoidable. For instance, it is doubtful whether the Muskogee Reservoir, constructed by the writer in 1912, should properly be included in the tables accompanying the paper. This reservoir consisted of thirty-six bays of the Ambursen type of dam, 30 ft. high, which formed substantially a circular reservoir, the bottom of which was fully paved with 6 in. of reinforced concrete, so that the element of creep on the stability of the structure, carries little significance, notwithstanding the fact that it was built almost entirely above the ground, the foundation consisting of sand, gravel, and cobbles. After more than twenty years' of use, the reservoir is still giving satisfactory service, and the leakage is scarcely measurable.

Mr. Lane's comment on the value of weep-holes or drains is useful and along the lines of the writer's experience. Mr. Koenig did not lay sufficient emphasis on the value of these drains for relieving upward pressure and reducing the danger against piping and this point was emphasized by the writer at the time.

The great strides which have been made in the field of soft foundation dams is evidenced by a comparison of Mr. Koenig's paper and the present work by Mr. Lane, who deserves great credit for the painstaking and thorough study he has made of existing dams. It was time that some courageous engineer should have attacked the ratios suggested by Bligh and Griffith which originated from an experience covering a limited variety of foundation materials. It has been almost an impossibility to proportion or amend these ratios intelligently for completely different types of materials. Furthermore, they were derived from contact with what to-day are rather small head developments. When such arbitrary ratios are applied to the modern soft-foundation dams of considerable height and great economic importance, they are likely to burden the development with extravagant cost.

The recommendation for a value of one-third for horizontal creep (even if it merely affords a justification to cut down previously recommended ratios to reasonable figures commensurate with modern practice), will be of service to the Engineering Profession. It points the way to reduced construction costs and will affect, favorably, the future storage and power developments that are close to the border line of economic feasibility.

Mr. Koenig's useful and studious paper and the discussions thereon, marked a summing up of the meager engineering experience with sand foundations that was available in 1911. It is well that Mr. Lane has had the time and patience to summarize the present state of the art. Mr. Koenig was one of the earliest engineers to direct attention to the great bearing value of a saturated sand, provided it could be confined adequately; he was also among

the first to recommend the use of a stilling pool for absorbing the kinetic energy of spillway discharge. While his conception of adequate stream-bed protection below the stilling pool may have been defective, it was a useful attempt, nevertheless, to start the engineer's thoughts along the proper channel. Mr. Lane properly advises great care against erosion below the spillway with resulting sudden drop in percolation resistance, the cause of many failures.

It would have been desirable to incorporate in the paper more data on dam failures. The writer has frequently had occasion to refer to the records of failures and finds a surprising lack of uniformity of opinion as to their causes, particularly with respect to soil foundations. The application of a study of soil mechanics to a determination of the value of percolation paths, while desirable, would be difficult to develop owing to the non-homogeneity of the materials at most dam sites.

In the hands of experienced designers, the writer believes that Mr. Lane's ratios can be lowered even more, but agrees that in this respect it is better to advance slowly.

ROBERT E. KENNEDY,[62] M. AM. SOC. C. E. (by letter).[62a]—Many small earth dams have been built or proposed in the Western Dakota and Eastern Montana area for drought and unemployment relief, erosion control, stock water, and general water conservation.

Constructed largely with no special compacting equipment permitted and frequently under the supervision of young and inexperienced college graduates, otherwise unemployed, the chief protection against percolation through the dam is an impervious core-wall, usually of wood. Table 3 of the paper gives a ready means of determining the necessary penetration of this core-wall, if the computation is based upon the shortest possible path, which is always shorter than the probable path, as shown in Fig. 11.

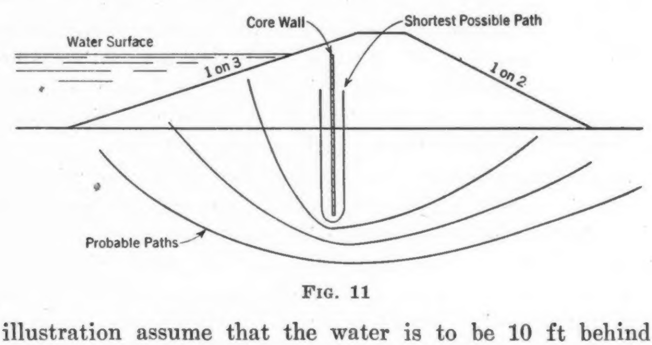

FIG. 11

For illustration assume that the water is to be 10 ft behind an earth dam placed on a coarse sandy foundation. The factor is 5. The minimum path should be 50 ft. If the earth is so loose as to have little value in retarding percolation the core-wall would extend 25 ft into the foundation, making a total height of 35 ft.

[62] State Engr., Bismarck, N. Dak.

[62a] Received by the Secretary April 10, 1935.

If the material in the dam is likely to be of the same general classification as that in the foundation, and if it is likely to receive a similar degree of compaction during construction, the penetration need be only 15 ft, making a total height of 25 ft.

For the ordinary soils with the usual methods of construction it may be considered safe to assume that the material in the dam will be about one-half as compact and dense as the material in the foundation. Then the necessary penetration in the illustration would be 20 ft.

For the benefit of those who would like to use a formula the following may serve, if used with discretion: $d = \frac{h}{2}(f - 1)$, in which d is the desired penetration; h is the depth of water behind the dam; and f is the suitable factor selected from Table 3.

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DISCUSSIONS

AN ASYMMETRIC PROBABILITY FUNCTION

Discussion

BY ARTHUR W. KEMPERT, ESQ.

ARTHUR W. KEMPERT,[49] ESQ. (by letter).[49a]—This interesting paper raises several issues which it would be desirable to discuss somewhat in detail. In the first place, the paper bears, in parts, a close resemblance to the earlier works by Thiele, Gram, Jorgensen, and Fisher. Secondly, it raises the moot questions of small samples and chance errors in relation to various pseudo-graphical methods. Last, but not least, there arises the important question as to whether or not the entire problem of the statistical analysis of rainfall, instead of being viewed as a homogeneous (or homograde) mass phenomenon, might not better be thought of as a heterogeneous object and analyzed by the aid of compound frequency curves.

A point very properly emphasized by Professor Slade is that "the use of probability paper and other graphical devices is an undesirable practice," especially when drawn to logarithmic or semi-logarithmic scales. The practice of a graphical process, no matter how carefully and judiciously it may be performed, is not reliable as a rule when applied to a limited number of observations or to a small sample. Investigators who have occasion to use statistical methods should be extremely cautious, therefore, lest they be led astray by the broad claims for some of the graphical schemes as a general procedure for the analysis of small samples. A case in point is the paper by R. D. Goodrich, M. Am. Soc. C. E. (cited by the author) in which the assertion is made[50] that the graphical method proposed "is developed especially for the examination of records consisting of from about 10 to 50 observations."

As an illustration of the danger of applying graphical methods to small samples of observations, the following data relating to some hitherto unpublished investigations on yeast wine cultures may be of interest.

NOTE.—The paper by J. J. Slade, Jr., Esq., was published in October, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: January, 1935, by Messrs. Gordon P. Williams and H. Alden Foster; February, 1935, by Messrs. R. D. Goodrich, and F. T. Mavis; March, 1935, by L. Standish Hall, Assoc. M. Am. Soc. C. E.; and April, 1935, by Arne Fisher, Esq.

[49] Statistical Asst. and Draftsman, Western Union Telegraph Co., New York, N. Y.

[49a] Received by the Secretary March 5, 1935.

[50] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 42.

A series of 149 samples, each containing 50 individual observations, was taken from certain pure yeast cultures present in a blend of high-grade vintages. The statistical object in question was found to vary from an observed low value of 8 to a high value of about 27. Table 19 gives the frequency distribution in: (1) The 32d sample (chosen at random); (2) a similar distribution for the 87th sample (also chosen at random); and (3) the frequency distribution of all 149 samples combined. If the 32d sample is fitted graphically, choosing the lower limit[51] at 10, the following equation is obtained:

TABLE 19.—FREQUENCY DISTRIBUTION, VARIATIONS IN PURE YEAST CULTURES IN A BLEND OF HIGH-GRADE WINES

Intervals of the variate (1)	Sample No. 32 (2)	Sample No. 87 (3)	149 samples combined (4)	Intervals of the variate (1)	Sample No. 32 (2)	Sample No. 87 (3)	149 samples combined (4)
8 to 10..........	0	0	5	18 to 20.........	8	12	1 644
10 to 12..........	2	0	78	20 to 22.........	3	5	846
12 to 14..........	8	0	615	22 to 24.........	1	2	270
14 to 16..........	15	17	1 742	24 to 26.........	0	1	52
16 to 18..........	13	13	2 195	26 to 28.........	0	0	3
Total.........	..	..	.....	....	50	50	7 450

$$t : 100 = 1 - (10)^{-0.0000224\,(R-10)^{2.355}} \ldots\ldots\ldots\ldots\ldots (85)$$

When fitted to a Gram series:

$$G(t) = 50\,[\phi_0\,(t) - 0.048\,\phi_3\,(t)] \ldots\ldots\ldots\ldots\ldots (86)$$

with $t = (x - 16.2) : 2.624$. The mean error of c_3 (or of $-$ 0.048) equals 0.057 and shows that no reliance can be placed on the empirically determined skewness in the small sample of 50. The mean error in the arithmetic mean exceeds 0.7 and the mean error in the dispersion (standard deviation) is greater than 0.3.

The situation is quite different in the case of the combined sample of 7 450 individual observations. This sample can be fitted to a Gram series of the following order (expressed in definite integral form):

$$G(x) = 7\,450 \left[\int_{-\infty}^{t} \phi_0\,(t)\,dt - 0.045\,\phi_2\,(t) \right] \ldots\ldots\ldots (87)$$

in which $\phi_0\,(t)$ is the generating function and $\phi_2\,(t)$ its second derivative, with $t = (x - 17.31) : 2.68$, which results in the frequency distribution given in Table 20, Columns (1) to (5).

The empirically determined skewness of $-$ 0.045 has a mean error of only 0.0047, about one-tenth of the magnitude of c_3, which is an indication of its reliability in contradistinction to the small sample of 50 in which the mean error of c_3 was greater than c_3 itself.

[51] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 4, Equation (*A*).

TABLE 20.—FREQUENCY DISTRIBUTIONS FOR A COMBINED SAMPLE OF 7 450 OBSERVATIONS

Values of x (1)	$G(x)$ (2)	$\Delta G(x)$ (3)	Observed data (4)	Intervals (5)	Gram (6)	Goodrich (7)	Type I (8)	Type II (9)	Type III (10)	Compound (11)	Observed (12)
<10	12	12	5	<10 to 10	0.001	0.000	3	0	0	3	5
12	107	95	78	10 to 12	0.013	0.058	76	2	0	78	78
14	764	657	615	12 to 14	0.088	0.205	569	45	2	616	615
16	2 451	1 687	1 742	14 to 16	0.226	0.285	1 388	334	27	1 749	1 742
18	4 648	2 197	2 195	16 to 18	0.295	0.243	1 110	911	171	2 192	2 195
20	6 278	1 630	1 644	18 to 20	0.219	0.138	291	911	447	1 649	1 644
22	7 083	805	846	20 to 22	0.108	0.054	24	334	489	847	846
24	7 363	280	270	22 to 24	0.038	0.014	1	45	223	269	270
26	7 437	74	52	24 to 26	0.010	0.003	0	2	42	44	52
28	7 450	13	3	26 to 28	0.002	0.000	0	0	3	3	3
Totals.	.....	7 450	7 450	....	1.000	1.000	3 462	2 584	1 404	7 450	7 450

A comparison between the relative frequencies of $\Delta G(x)$ and the corresponding relative frequencies for the small sample (No. 32) as fitted to the Goodrich formula,[51] is given in Columns (7) and (8), Table 20. The two curves are unlike and demonstrate the danger inherent in any attempt to evaluate a presumptive frequency curve from a small sample.

In view of the close correspondence between the 7 450 observations and the Gram frequency curve, many investigators are likely to conclude from purely empirical considerations that the material in question is homogeneous; such an inference, however, would be unwarranted. With the aid of experimental laboratory analysis, based on the concept of Johannsen's "pure lines" and his own theory of probability synthesis, Mr. Arne Fisher[52] has demonstrated that the observed mass phenomenon, instead of being of a single type, actually is composed of three distinctive types, namely, a Bordeaux vintage (Capestang), a Burgundy vintage (Beaune type), and a vintage from the Crimean Peninsula, characterized by the statistical parameters given in Table 21. This leads to a compound frequency curve, such as shown in Table 20, Columns (8) to (12).

TABLE 21.—COMPOUND DISTRIBUTION OF THREE TYPES

Type	Name	Area	Mean	Dispersion	Skewness
I..................	Capestang........	3 462	15.6	1.800	None
II..................	Beaune...........	2 584	18.0	1.910	None
III..................	Crimea...........	1 404	20.2	2.060	None
Total compound..........	....	7 450	17.306	2.664	—0.045

A few remarks seem necessary in regard to the difficult problem of the calculation of mean (or probable) errors of the frequency constants. In the case of the regular Gram series this problem offers no serious difficulty because of the orthogonal properties of the individual terms of the series

[52] "Frequency Curves," by Arne Fisher, New York, Macmillan Co., 1922.

which, in the terminology of Thiele, makes the constants, *c*, mutually free (independent or uncorrelated) of each other.

In the case of the logarithmically transformed frequency functions as described by Professor Slade, and also in the case of a graphical process, there arises, however, the formidable obstacle that the constants are not mutually free, but are bound or correlated, in the sense that their values will depend upon the lower (or upper) limits of the variate. Any change whatever in the positions of these limits will influence the other constants—the *c* and the *d*, in Professor Slade's formula, or in other formulas that depend on lower (or upper) limits.

Because of the fundamental fact that the constants are bound or correlated, it would seem, therefore, that prior attempts[53] to evaluate the mean (or probable) errors largely must be looked upon as failures. The fundamental implications in such proofs are that the constants are not correlated. Proponents of graphical methods admit that "it does not seem possible to derive any formula for the probable error of the index, *c*,"[54] and, in another place, in discussing the errors in the upper and lower limits, it is also stated that "it does not seem possible to derive any formulas applicable to them."[55] Admissions such as those just quoted naturally lead one to regard many graphical methods in a light of skepticism, as it were.

As one of his requirements, Professor Slade states that "the curve must be specifiable completely by moments no higher than the third," and for higher moments "it becomes necessary to search for physical substitutes." This sound advice seems to be in complete accord with the following rules by Thiele[56]: (1) For first and second semi-invariants (or moments) rely exclusively on the observations; (2) for semi-invariants of higher order than the sixth rely exclusively on theory; and (3) for intermediate semi-invariants (or moments) rely partly upon theory and partly upon the observations.

In conclusion, it may be said that some slight inconsistencies have occurred in the author's treatment of his subject. In one place he states that graphical methods "convey to the eye a simplicity which does not really exist," yet under Example *b*, in comparing his own results with those of Fisher, the author employs a rather crude graphical method of comparison. As matters turn out, Professor Slade's graduated curve in Fig. 1 does not provide nearly as good a fit as Fisher's graduation when subjected to Pearson's test of "goodness of fit."

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[53] As, for example, *Transactions*, Am. Soc. C. E., Vol. 91 (1927), pp. 97–102.
[54] *Loc. cit.*, p. 99.
[55] *Loc. cit.*, p. 100.
[56] "The Mathematical Theory of Probabilities," by Arne Fisher, p. 217.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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DISCUSSIONS

ANALYSIS OF CONTINUOUS STRUCTURES BY TRAVERSING THE ELASTIC CURVES

Discussion

BY FANG-YIN TSAI, ASSOC. M. AM. SOC. C. E.

FANG-YIN TSAI,[21] ASSOC. M. AM. SOC. C. E. (by letter).[21a]—The idea behind this method is certainly novel and ingenious. However, the writer does not consider that it furnishes "a quick and easy solution" for continuous structures, as claimed by the author.

In illustrating the application of the method, Mr. Stewart has intentionally selected several unusually simple cases most of which are rarely, if ever, met in actual practice. Undoubtedly, this was done merely to demonstrate the simplicity of the procedure, but it creates the erroneous impression that the same simple procedure can be applied also to any continuous structure in general. For instance, the simple solution of the two-legged bent in Fig. 3 does not apply when the beam and columns do not have the same length, or when the loading is not symmetrical. It may be also noted that the solution for the two-legged bent in Fig. 4 is somewhat lengthy, and does not seem as simple and direct as that by the slope-deflection method. Even for the extremely simple case of the continuous beam (see Fig. 6), the author's claim that the solution for all moments is quick and easy, does not seem to be justified. For a simple case, such as this one, the solution by the graphical method of fixed points, or conjugate points, will be found to be the quickest and easiest of all.

The method illustrated in Fig. 5 evidently applies only for the particular case of yielding supports and its application to continuous structures will be found very difficult if not impossible. Although Mr. Stewart states that "settlement of supports * * * can be incorporated in the geometry of a traverse", he has not explained how this can be done. This problem could be

NOTE.—The paper by Ralph W. Stewart, M. Am. Soc. C. E., was published in October, 1934, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: December, 1934, by Messrs Garrett B. Drummond, Austin H. Reeves, E. G. Paulet, Adolphus Mitchell, and David M. Wilson; and March, 1935, by Messrs. W. H. Kirkbride, R. B. Ketchum, A. Floris, and Ivan M. Nelidov.

[21] Prof., Dept. of Civ. Eng., National Tsing Hua Univ., Peiping, China.

[21a] Received by the Secretary, January 16, 1935.

solved readily by Equation (3) and Fig. 2, from which it is seen that the traverse must also pass through the supports after they have settled. For illustration, the writer has solved the problem in Fig. 7, assuming that Supports B and C have settled by an amount of $d_1 = 0.2$ in. and $d_2 = 0.1$ in., respectively, relative to the elevation of Supports A and D. An 18-in 47-lb steel I-beam is assumed, with a moment of inertia, $I = 768.6$ in.[4] and a modulus of elasticity, $E = 30\,000\,000$ lb per sq in. The traverse is shown in Fig. 10 (c) for which the equation is,

$$\theta_1 - A_1 + \Delta_1 + \Delta_2 - A_2 + \Delta_3 + \Delta_4 - A_3 + \theta_4 = 0 \ldots\ldots (50)$$

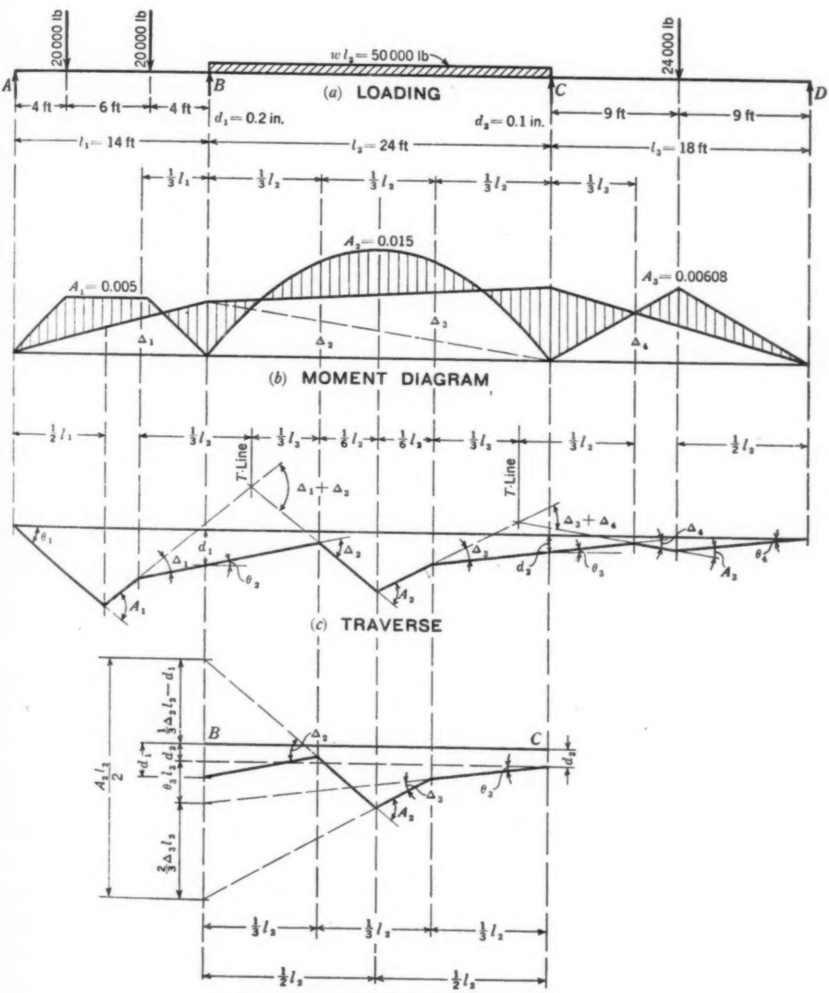

FIG. 10.

From the geometry of the moment diagram and the traverse, the following relations can be found:

$$\Delta_2 = \frac{l_2}{l_1}\,\Delta_1 \qquad (51)$$

$$\Delta_3 = \frac{l_2}{l_3}\,\Delta_4 \qquad (52)$$

$$\theta_1 = \frac{A_1}{2} - \frac{\Delta_2}{3} + \frac{d_1}{l_1} \qquad (53)$$

and,

$$\theta_4 = \frac{A_3}{2} - \frac{\Delta_3}{3} + \frac{d_2}{l_3} \qquad (54)$$

Substituting Equations (51) to (54) and the various numerical values (noting that areas of the $\frac{M}{EI}$-diagrams due to loading are: $A_1 = 0.005$, $A_2 = 0.015$, and $A_3 = 0.00608$) in Equation (50),

$$-\,0.1889 + 2.381\,\Delta_1 + 2\,\Delta_4 = 0 \qquad (55)$$

Another equation may be obtained from the goemetry of the traverse in the center span (Fig. 10 (*d*)) as follows:

$$\frac{2}{3}\,\Delta_3\,l_2 + \theta_3\,l_2 + d_2 + \frac{1}{3}\,\Delta_2\,l_2 - d_1 - \frac{1}{2}\,A_2\,l_2 = 0 \qquad (56)$$

Noting that $\theta_3 = -\frac{A_3}{2} + \frac{2\,\Delta_4}{3} + \frac{d_2}{l_3}$, the following equation is obtained after proper substitution:

$$-\,0.1043 + 0.571\,\Delta_1 + 1.556\,\Delta_4 = 0 \qquad (57)$$

Solving Equations (55) and (57) simultaneously, $\Delta_1 = 0.00331$; $M_B = -\,76\,000$ ft-lb; $\Delta_4 = 0.00551$; and $M_C = -\,98\,000$ ft-lb.

The writer has also solved the same problem by the equation of three moments and the slope-deflection method, both of which are found much more simple than the method proposed in this paper. This seems quite natural since, in applying the method, the equations must be formed from the geometry of the traverse instead of by writing the slope-deflection or three-moment equations directly, to which the author objects. The writer would appreciate a suggestion for simplifying the solution of the foregoing problem.

In the "Conclusions" of the paper, Mr. Stewart states that the key constants of the analysis are simple $\frac{M}{EI}$-diagrams and cantilever $\frac{M}{EI}$-diagrams. It is evident that he has neglected entirely the centers of gravity of the diagrams, which are absolutely necessary, as emphasized by the author

himself (see Principle (2)). This oversight is natural since the loading in all his illustrations is symmetrical for every span and, consequently, all the centers of gravity of the simple moment diagrams are known to be at the center of every span.

In discussing the application of the method to the case of variable moment of inertia, Mr. Stewart mentions the tables presented by Walter Ruppel,[6] Assoc. M. Am. Soc. C. E., and states that lists of $\frac{M}{EI}$ - areas for simple moments are unknown to him. The simple moment areas for various loading, as well as their centers of gravity, are given in Mr. Ruppel's tables. Elsewhere,[22] the writer has derived the relations between the various $\frac{M}{EI}$ - diagrams and the coefficients in Ruppel's tables, as follows (see Fig. 11):

$$A_L = \frac{p\,l}{I'} \qquad (58)$$

$$A_R = \frac{q\,l}{I'} \qquad (59)$$

$$A_0 = \frac{W\,l^2}{I'}(sp + tq) \qquad (60)$$

$$u = u \qquad (61)$$

$$v = v \qquad (62)$$

and,

$$g = \frac{tq}{sp + tq} \qquad (63)$$

in which, A_L = area of $\frac{M}{I}$-diagram due to a moment of unity applied at the left end of a simple beam; A_R = area of $\frac{M}{I}$ - diagram due to a moment of unity applied at the right end of a simple beam; A_0 = area of $\frac{M}{I}$ - diagram due to loading on a simple beam; ul = abscissa of the center of gravity of A_L from the left support, L, of a simple beam; vl = abscissa of the center of gravity of A_R from the right support, R, of a simple beam; gl = abscissa of the center of gravity of A_0 from the left support, L, of a simple beam; I' = minimum moment of inertia of beam; W = total load on span; l = span length; p, q, u, and v = beam coefficients in Ruppel's tables, which depend on the shape of the beam only; and s and t = load coefficients in Ruppel's tables, which depend on both the shape of the beam and the type of loading.

[6] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), pp. 167–187.

[22] "Theorem of Three Moments in General Form," The Science Repts, National Tsing Hua Univ., Peiping, China, Series A, Vol. II, pp. 19–36, April, 1933.

In Equations (58) to (63), the modulus of elasticity, E, has been omitted since it is always assumed to be constant for the entire structure. It may be noted that, for a beam with a moment of inertia varying in any manner, the following relations are always valid:

$$pu = qv \qquad (64)$$

and,

$$A_L u = A_R v \qquad (65)$$

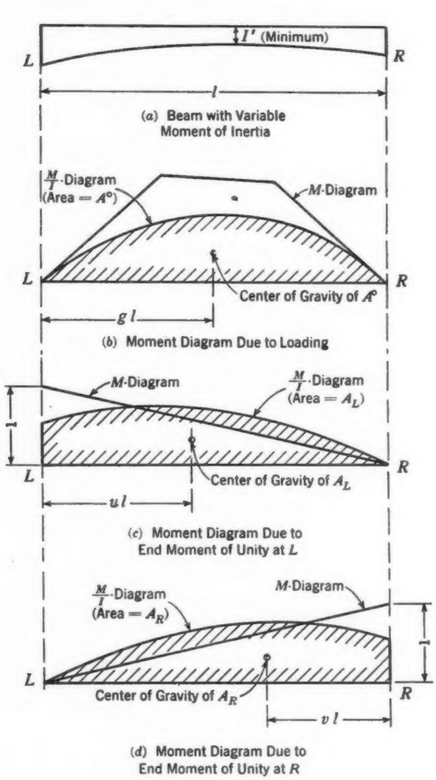

FIG. 11.

which follow directly from Maxwell's theorem of reciprocal deflections (angular). Therefore, to analyze, by any method, a continuous structure with a moment of inertia varying in any manner, five independent constants or coefficients must be known (three beam coefficients and two load coefficients) for every span of the structure considered as a simply supported beam; and these five coefficients may be expressed in various ways to suit any particular method of analysis. With the aid of Equations (58) to (63), Ruppel's tables can be utilized in applying the method to the case of variable moment of inertia.

Mr. Stewart also states that,

"From Maxwell's theorem, the final distortion of the beam from the combined action of the external forces may be treated as the sum of the distortion, due to each force acting separately."

As far as the writer knows, the statement is just one aspect of the general principle of superposition, which has not been attributed to Maxwell. The writer has also searched in the two volumes of "Maxwell's Scientific Papers"[23] and has failed to find anything like the theorem explicitly stated therein.

[23] Pub. by Librairie Scientifique J. Hermann, Paris, by arrangement with the Cambridge Univ. Press, 1890.

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DISCUSSIONS

RELATION BETWEEN RAINFALL AND RUN-OFF FROM SMALL URBAN AREAS

Discussion

BY MESSRS. FRANKLIN F. SNYDER, MERRILL M. BERNARD, AND LEROY K. SHERMAN

FRANKLIN F. SNYDER,[6] JUN. AM. SOC. C. E. (by letter).[6a]—Figs. 15 and 16 of the paper, giving basic data for typical storms on the three areas investigated, provide opportunity for various instructive studies. Computed run-off is obtained by applying the run-off factor to the 100% run-off rates as obtained by the unit-graph distribution of the rainfall. The writer used "net rainfall" and the author's unit graphs to study the same storms. "Net rainfall" (which, in amount, is equal to the run-off) is defined as precipitation minus losses, of which infiltration is much the largest and will be termed in this discussion "infiltration loss."

In accordance with the line of reasoning advanced by R. E. Horton, M. Am. Soc. C. E.,[7] a constant infiltration rate was determined for each storm. The constant infiltration rates for the three storms are, Area *A*, rain of September 8, 1926, 0.61 in. per hr; Area *B*, rain of August 27, 1921, 1.28 in. per hr; and Area *C*, rain of September 8, 1920, 0.45 in. per hr. Subtracting these rates of infiltration from the given precipitation rates minute by minute gave net rainfall rates, which were then distributed by the unit graph for the particular area to obtain computed run-off rates.

The computed run-off rates were then compared with the observed rates and the differences or errors obtained for each minute, from which, in turn, were computed the probable errors. For the storm of September 8, 1926, on Area *A*, the probable error for any one minute was 0.14 cu ft per sec, as compared with an approximate probable error, reported in the paper, of 0.21 cu ft per sec. This value and others given for the authors' method

NOTE.—The paper by W. W. Horner, M. Am. Soc. C. E., and F. L. Flynt, Assoc. M. Am. Soc. C. E., was published in October, 1934, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[6] Junior Engr., U. S. Geological Survey, Washington, D. C.

[6a] Received by the Secretary February 9, 1935.

[7] "The Rôle of Infiltration in the Hydrologic Cycle," by R. E. Horton, M. Am. Soc. C. E., *Transactions*, Am. Geophysical Union, National Research Council of the National Academy of Science, June, 1933, p. 446.

are only approximate, as the differences between computed and observed rates of run-off were scaled from Fig. 15. The probable error for the rain of August 27, 1921, on Area *B*, was found to be 0.19 cu ft per sec for any one minute, as against 0.08 cu ft per sec for the results given in the paper. For Area *C*, and the rain of September 8, 1920, the corresponding probable errors were 0.21 and 0.18 cu ft per sec, respectively.

This comparison indicates a close check of the degree of accuracy reported in the paper, although the run-off factor gives better results in two cases out of three. Possibly, improvements would be obtained by using a variable rate of infiltration loss and a variable run-off factor.

According to Mr. Horton,[7] the infiltration capacity of a natural soil is high when the rain begins, decreases rapidly at first, and approaches stability within an interval ranging commonly from 1 hr to 3 hr. If this is true a constant infiltration rate might not have been reached in the three storms studied, as they lasted less than 1 hr each. The writer's studies bear out the theory of a variable infiltration rate, although they do not show clearly the high rates of loss at the beginning of the rain with a graduated decline thereafter.

A study of the algebraic sign and variation of the differences between computed and observed rates of run-off, and trials using a variable rate of infiltration, gave the following results for the three storms: The infiltration rate varied directly as the intensity of precipitation, but no attempt was made to determine the exact order of its variation. The approximate average rate of infiltration for 5-min periods for Area *A* ranged from about 1.0 in. per hr at the beginning of the storm (10:13 A.M.) to 0.6 in. per hr at 10:40 A.M., and decreased to about 0.5 in. per hr at 11:00 A.M. In Area *B* the beginning rates (average for 5-min intervals) were about 0.7 in. per hr, and these increased to 1.8 in. per hr at the end of the storm, when high intensity of rainfall occurred. In Area *C* an average infiltration rate of approximately 0.4 in. per hr continued for practically the entire storm, except for the two periods of high rainfall intensity, when it approached 0.5 to 0.8 in. per hr.

In all four peaks of the three storms the use of the average infiltration rates gave higher peak run-off rates than the observed rates, whereas the use of the run-off factor gave computed results lower than the observed rates for the higher peaks and higher than the observed rates for the two lesser peaks. The use of a rate of infiltration varying with precipitation, therefore, would bring the computed run-off rates nearer the observed rates in all four peaks, whereas a varying run-off factor would improve two of the computed results and add to the error of the other two. Accordingly, it appears that the use of a variable infiltration rate based on variations in precipitation intensity would give more consistent results than the use of a variable run-off factor.

Moreover, for all four peaks the computed run-off rates by the average infiltration method were larger than those obtained by the run-off factor—

a difference which is inherent in the two methods. This is the result of subtracting a constant amount from the minute-by-minute rainfall, thus increasing the percentage of run-off from the higher rates.

Having an abridged copy of the original manuscript available,[7a] the writer compared average infiltration rates for the same storm (that of September 8, 1920) on the three areas. The results are given in Table 12. Column (5) shows the loss in inches per hour on an area that is 100% pervious, under the assumption of a straight-line relation between area of pervious ground and the quantity of infiltration. Comparison of Columns (3) and (5), Table 12, shows an apparent variation of average infiltration rate with intensity of rainfall.

TABLE 12.—RAIN OF SEPTEMBER 8, 1920

Area (1)	Percentage of area pervious (2)	Mean intensity of rainfall, in inches per hour (3)	Average loss, in inches per hour (4)	Ratio, Column (4) / Column (2) (5)
A	50	2.63	1.06	2.12
B	70	1.89	0.77	1.10
C	28	2.43	0.45	1.61

The authors have presented a comprehensive analysis of a valuable record of hydrologic data. The data and results are invaluable to the designer of sewers, and the basic data of the several storms provide an opportunity for the development and testing of methods of analysis of rainfall and run-off relations.

MERRILL M. BERNARD,[8] M. AM. SOC. C. E. (by letter).[8a]—Injecting modern thought into the science of storm-sewer design, the authors have given an interesting chronicle of an unusual experience in hydrologic research. It is particularly interesting to note the general applicability of the unit-hydrograph method to conditions ranging from those prevailing on small urban inlet areas of only 2 or 3 acres, with changes in rate of recorded rainfall at 1-min intervals, to those prevailing in stream basins covering several thousand square miles, utilizing U. S. Weather Bureau data in which rainfall is recorded at 24-hr intervals.

An absence of unit-time (1-min) records of rainfall in the studies recorded in this paper made it necessary for the authors to arrive at the factors fixing the shape and extent of their unit graphs "from a study of the records of a few short rains of fairly uniform intensities." That their method and the resulting equations are successful is demonstrated throughout the work and is one of the highlights of the paper. The resulting unit graph, however, is the same as that presented by LeRoy K. Sherman,[9] M. Am. Soc. C. E., being

[7a] Original manuscript on file at Engineering Societies Library, 29 West 39th Street, New York, N. Y.

[8] Cons. Civ. Engr., Crowley, La.

[8a] Received by the Secretary February 9, 1935.

[9] "Streamflow from Rainfall by Unit-Graph Method," *Engineering News-Record*, April 7, 1932.

the hydrograph of 0.0167 in. of run-off depth (comparable to 1 in. of run-off depth from a 24-hr rainfall), applied throughout 1 min of rainfall at the rate of 1 in. per hr.

The writer has suggested the "distribution graph,[10] which shows the proportion of the run-off from any unit-time rainfall throughout the period of run-off, expressed as percentage of the total run-off. The conversion of the unit graph to the distribution graph, when rainfall is expressed in rate or in depth per unit of area, is accomplished by moving the decimal point two places to the right.

Table 13 constitutes a demonstration of the general application of the unit-hydrograph theory (on which Part I of the paper is based), applying methods developed on large basins to the rainfall of September 8, 1920, on Area *C*, shown in Fig. 15. In this case rainfall, expressed as rate in inches per hour, is reduced to inches of depth per minute and is recorded in Column (2), which makes it comparable to daily rainfall.

The unit graph for the area has been computed from Equations (1), (2), and (4) of the paper and the values for depth have been converted into the percentages of the distribution graph (see Column (3), Table 13). Rainfall (or, with the assumption of no loss, the theoretical 100% run-off) is distributed throughout the period of run-off by applying the 1-min figures of the distribution graph to rainfall depth, as shown in Columns (4) to (18), Table 13. Distributed rainfall (or 100% run-off) is now accumulated horizontally, giving the results for the author's theoretical 100% run-off graph, or the writer's pluviagraph (Columns (19) and (20), Table 13). The writer has distributed the figures computed for the unit graph beyond 14 min, throughout the distribution graph, accounting for the slight differences between the values in Column (22), Table 13, and those in Fig. 15.

As the storm of September 8, 1920 (Fig. 15), reached a peak in its later stages, it will be used to demonstrate the effect of the distribution of rainfall intensities throughout the period of rainfall on the 100% run-off or pluviagraph results. If the rainfall rates are arranged in order of magnitude, thereby assuming that the storm reached a peak in the first minute of rainfall, and that it was distributed as previously demonstrated (see Column (21), Table 13), the pluviagraph peak will shift from 5.33 ppm to 5.17 ppm without any appreciable change in value. It can be shown, however, that the most critical arrangement of intensities is that in which the high intensities are grouped around the minute that marks the peak of the distribution graph, which is, for the example given, the 6th minute of rainfall. Thus, on arranging rainfall and distributing it as previously explained (Column (22), Table 13), the pluviagraph peak shifts to 5.21 ppm and gives a result 12% greater than that developed by the storm itself.

The rate ratio, $P = \frac{\Sigma Q_t}{\Sigma Q_c}$, mentioned in connection with Fig. 17 of the

[10] "An Approach to Determinate Stream Flow," *Proceedings*, Am. Soc. C. E., January, 1934, p. 5.

TABLE 13.—Application of Distribution Graph for Area *C* (Fig. 25)

Time, in minutes	Rainfall depth, in inches	Distribution graph (percentages)	Rainfall (or 100% run-off) distribution	Run-off (100%) Accumulation, in Inches															Theoretical 100% Run-off, or Pluviagraph		Inches per Hour Re-arranged In:	
(1)	(2)	(3)		(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	In. inches (19)	In. inches per hour (20)	Order of magnitude (21)	Critical order (22)
5:11	0.010	0.8		0.0001	......	......	......	......	......	......	......	......	......	......	......	......	......	......	0.0001	0.006	0.060	0.012
12	0.010	3.1		0.0003	0.0001	......	......	......	......	......	......	......	......	......	......	......	......	......	0.0004	0.024	0.276	0.060
13	0.020	7.1		0.0007	0.0003	0.0002	......	......	......	......	......	......	......	......	......	......	......	......	0.0012	0.072	0.756	0.198
14	0.005	12.6		0.0013	0.0007	0.0006	0.0000	......	......	......	......	......	......	......	......	......	......	......	0.0026	0.156	1.578	0.486
15	0.005	19.7		0.0020	0.0013	0.0014	0.0002	0.0000	......	......	......	......	......	......	......	......	......	......	0.0049	0.294	2.796	0.936
16	0.020	28.3		0.0028	0.0020	0.0025	0.0004	0.0002	0.0002	......	......	......	......	......	......	......	......	......	0.0081	0.486	4.464	1.728
17	0.020	14.2		0.0014	0.0028	0.0039	0.0006	0.0004	0.0006	0.0002	......	......	......	......	......	......	......	......	0.0099	0.594	4.854*	2.574
18	0.060	7.0		0.0007	0.0007	0.0056	0.0010	0.0006	0.0014	0.0006	0.0005	......	......	......	......	......	......	......	0.0111	0.666	4.590	3.414
19	0.070	3.5		0.0004	0.0004	0.0028	0.0014	0.0010	0.0025	0.0014	0.0019	0.0006	......	......	......	......	......	......	0.0124	0.744	4.080	4.254
5:20	0.050	2.0		0.0002	0.0002	0.0014	0.0007	0.0014	0.0039	0.0025	0.0043	0.0022	0.0005	......	......	......	......	......	0.0173	1.038	3.630	5.046
21	0.030	0.9		0.0001	0.0001	0.0007	0.0003	0.0007	0.0056	0.0039	0.0076	0.0050	0.0016	0.0002	......	......	......	......	0.0258	1.548	3.240	5.370*
22	0.020	0.5		......	0.0001	0.0004	0.0002	0.0003	0.0028	0.0056	0.0118	0.0088	0.0035	0.0009	0.0002	......	......	......	0.0345	2.070	2.940	4.950
23	0.040	0.2		......	......	0.0002	0.0001	0.0002	0.0014	0.0028	0.0170	0.0138	0.0063	0.0021	0.0006	0.0004	......	......	0.0449	2.694	2.622	4.080
24	0.020	0.1		......	......	0.0001	0.0001	0.0001	0.0007	0.0014	0.0085	0.0198	0.0098	0.0038	0.0014	0.0012	0.0002	......	0.0471	2.766	2.220	3.270
25	0.050			......	......	0.0001	......	0.0001	0.0004	0.0007	0.0042	0.0099	0.0141	0.0060	0.0025	0.0028	0.0006	0.0005	0.0421	2.526	1.842	2.652
26	0.023	100.0		0.0002	......	0.0001	......	......	0.0002	0.0004	0.0020	0.0048	0.0070	0.0085	0.0039	0.0050	0.0014	0.0016	0.0351	2.106	1.554	2.046
27	0.090	....		0.0007	0.0007	......	......	......	0.0001	0.0002	0.0012	0.0025	0.0035	0.0043	0.0056	0.0078	0.0025	0.0035	0.0326	1.956	1.386	1.620
28	0.110	....		0.0016	0.0028	0.0009	......	......	0.0001	0.0001	0.0005	0.0014	0.0017	0.0021	0.0028	0.0112	0.0039	0.0063	0.0354	2.124	1.278	1.398
29	0.120	....		0.0029	0.0064	0.0034	0.0010	......	0.0001	0.0001	0.0003	0.0006	0.0010	0.0009	0.0014	0.0056	0.0056	0.0098	0.0391	2.346	1.236	1.284
5:30	0.050	....		0.0045	0.0113	0.0078	0.0037	0.0005	......	0.0001	0.0001	0.0004	0.0005	0.0006	0.0007	0.0028	0.0028	0.0141	0.0499	2.994	1.200	1.224
31	0.020	....		0.0065	0.0177	0.0139	0.0085	0.0016	0.0002	......	0.0001	0.0001	0.0003	0.0003	0.0004	0.0014	0.0014	0.0070	0.0594	3.564	1.146	1.152
32	0.020	....		0.0033	0.0255	0.0217	0.0151	0.0035	0.0006	0.0002	......	0.0001	0.0001	0.0002	0.0002	0.0008	0.0007	0.0035	0.0755	4.530	1.056	1.062
				0.0016	0.0128	0.0310	0.0236	0.0063	0.0014	0.0006	......	......	0.0001	0.0001	0.0001	0.0004	0.0004	0.0017	0.0801	4.806*	0.918	0.924
				0.0008	0.0063	0.0156	0.0340	0.0098	0.0025	0.0014	......	......	......	......	0.0001	0.0002	0.0002	0.0010	0.0719	4.314	0.696	0.696
				0.0005	0.0032	0.0077	0.0170	0.0141	0.0039	0.0025	......	......	......	......	0.0001	0.0002	0.0001	0.0005	0.0498	2.988	0.528	0.528
				0.0002	0.0018	0.0039	0.0084	0.0070	0.0056	0.0039	......	......	......	......	......	0.0001	0.0001	0.0003	0.0312	1.872	0.366	0.366
				0.0001	0.0008	0.0022	0.0042	0.0035	0.0028	0.0056	......	......	......	......	......	......	0.0001	0.0001	0.0194	1.164	0.240	0.240
				0.0001	0.0004	0.0010	0.0024	0.0017	0.0014	0.0028	......	......	......	......	......	......	......	0.0001	0.0099	0.594	0.120	0.120
				......	0.0002	0.0006	0.0012	0.0010	0.0007	0.0014	......	......	......	......	......	......	......	......	0.0051	0.306	0.060	0.060
				......	0.0001	0.0002	0.0006	0.0005	0.0004	0.0007	......	......	......	......	......	......	......	......	0.0023	0.138	0.030	0.012
				......	......	0.0001	0.0002	0.0003	0.0002	0.0004	......	......	......	......	......	......	......	......	0.0012	0.072	0.012	0.006
				......	......	......	0.0001	0.0001	0.0001	0.0002	......	......	......	......	......	......	......	......	0.0005	0.030	0.006	......
				......	......	......	......	0.0001	0.0001	0.0001	......	......	......	......	......	......	......	......	0.0003	0.018	......	......
				......	......	......	......	......	0.0001	0.0001	......	......	......	......	......	......	......	......	0.0002	0.012	......	......
				......	......	......	......	......	......	0.0001	......	......	......	......	......	......	......	......	0.0001	0.006	......	......

* Peak values.

paper, has been termed the "retention coefficient" by the writer in his recent paper,[11] but which should more properly be called the "storm, or flood, coefficient."

Under the heading "Suggestions for Application to Sewer Design," the authors place conservative limits upon the immediate use of their work in actual design, holding forth the hope that future research will make it possible to give proper values to such factors as j, k, and t_1 for conditions as they are combined in the typical city block. As sewer design is usually based on an estimate of the effect of future growth on surface conditions, these factors, no doubt, can be classified and standardized, as the now widely used "coefficient of imperviousness" and the coefficient, C, of the rational method.

The unit-hydrograph method has other advantages than those demonstrated by this paper. Assume, for example, that the means are at hand to produce acceptable values for the factors, j, K, and t_1, on an urban drainage area consisting of four inlet areas of 10 acres each. All are assumed to be alike in character, and, therefore, all have the same unit and distribution graphs. Other conditions are, as follows: The frequency to be met by design is, once in 10 yr; inlets are spaced at 1 000-ft intervals along a proposed sewer of circular section, beginning with Inlet No. 1; the slope ratio of the sewer gradient is 0.002; Kutter's n is 0.013; and, the 10-yr rainfall intensity equation for the locality is,

$$i = \frac{182}{t + 23} \qquad \qquad (6)$$

A synthetic storm is developed by computing average rainfall rates for the various duration periods from Equation (6). These rates have been reduced to an average rate throughout any minute, and are the minute differences in the product of i and t-values, representing typical deviations from the averages defined by the intensity equation. Sound design demands the assumption of limiting conditions, and, therefore, rainfall is arranged in critical order (see Column (2), Table 14). It is recognized that this special arrangement of rainfall intensities tends to modify the conception of the storm as having a 10-yr frequency.

Retardation, through surface pondage, is reflected and accounted for in the distribution graph of the area, leaving the principal direct loss, particularly for rainfalls of short duration, to be that of infiltration, which is well expressed as a deduction, as suggested by Robert E. Horton, M. Am. Soc. C. E.[12] The problem here considered assumes an average infiltration loss of 0.50 in. per hr (Column (3), Table 14). The average rainfall excess, or that portion which appears as run-off, is given in Column (4).

The distribution graph for any one of the inlet areas is given in Column (5), Table 14. As developed on large basins, this graph represents the flow created by surface run-off from any rainfall, confined to a unit-time interval,

[11] "An Approach to Determinate Stream Flow," *Proceedings*, Am. Soc. C. E., January, 1934, p. 14.

[12] "The Rôle of Infiltration in the Hydrologic Cycle," *Transactions*, National Research Council, 1933, p. 450.

TABLE 14.—COMPUTATION OF RUN-OFF FROM INLET AREA

Elapsed time, in minutes	Average rainfall intensity, in inches per hour	Average rate of infiltration, in inches per hour	Average rainfall excess, in inches per hour	DISTRIBUTION GRAPH		Run-off distribution	RUN-OFF ACCUMULATION, IN CUBIC FEET PER SECOND					
				Percentage	Acres							
(1)	(2)	(3)	(4)	(5)	(6)		(7)	(8)	(9)	(10)	(11)	(12)
1....	3.80	0.50	3.30	1	0.10		0.33	....	....	....	....	....
2....	4.25	0.50	3.75	2	0.20		0.66	0.38	....	....	....	....
3....	4.88	0.50	4.38	3	0.30		0.99	0.75	0.44	....	....	....
4....	5.57	0.50	5.07	5	0.50		1.65	1.13	0.88	0.51	....	....
5....	6.43	0.50	5.93	10	1.00		3.30	1.88	1.31	1.02	0.59	....
6....	7.58	0.50	7.08	17	1.70		5.61	3.75	2.19	1.52	1.19	0.71
7....	6.99	0.50	6.49	14	1.40		4.62	6.38	4.38	2.54	1.78	1.42
8....	5.93	0.50	5.43	11	1.10		3.63	5.25	7.45	5.07	2.97	2.13
9....	5.12	0.50	4.62	9	0.90		2.97	4.13	6.14	8.62	5.93	3.54
10....	4.45	0.50	3.95	7	0.70		2.31	3.38	4.82	7.10	10.08	7.08
11....	4.00	0.50	3.50	6	0.60		1.98	2.63	3.94	5.58	8.31	12.04
12....	3.50	0.50	3.00	5	0.50		1.65	2.25	3.17	4.56	6.53	9.92
13....	3.30	0.50	2.80	4	0.40		1.32	1.88	2.62	3.55	5.33	7.79
14....	3.20	0.50	2.70	3	0.30		0.99	1.50	2.18	3.04	4.15	6.37
15....	3.00	0.50	2.50	2	0.20		0.66	1.13	1.75	2.53	3.56	4.95
16....	2.80	0.50	2.30	1	0.10		0.33	0.75	1.31	2.03	2.96	4.25
				100	10.00							
17....	2.70	0.50	2.20				....	0.38	0.88	1.52	2.37	3.54

Exposed time, in minutes	RUN-OFF ACCUMULATION, IN CUBIC FEET PER SECOND — (*Continued*)											Run-off measured at inlet, in cubic feet per second
(1)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
1....	....	....	....	....	....	....	....	.....	.....	.....	.....	0.33
2....	....	....	....	....	....	....	....	.....	.....	.....	.....	1.04
3....	....	....	....	....	....	....	....	.....	.....	.....	.....	2.18
4....	....	....	....	....	....	....	....	.....	.....	.....	.....	4.17
5....	....	....	....	....	....	....	....	.....	.....	.....	.....	8.10
6....	....	....	....	....	....	....	....	.....	.....	.....	.....	14.97
7....	0.65	....	....	....	....	....	....	.....	.....	.....	.....	21.77
8....	1.30	0.54	....	....	....	....	....	.....	.....	.....	.....	28.34
9....	1.95	1.09	0.46	....	....	....	....	.....	.....	.....	.....	34.83
10....	3.25	1.63	0.92	0.40	....	....	....	.....	.....	.....	.....	40.97
11....	6.49	2.71	1.39	0.79	0.35	....	....	.....	.....	.....	.....	46.21
12....	11.03	5.43	2.31	1.19	0.70	0.30	....	.....	.....	.....	.....	49.04
13....	9.09	9.24	4.62	1.98	1.05	0.60	0.28	.....	.....	.....	.....	49.35
14....	7.14	7.61	7.85	3.95	1.75	0.90	0.56	0.27	.....	.....	.....	48.26
15....	5.84	5.97	6.47	6.72	3.50	1.50	0.84	0.56	0.25	.....	.....	46.23
16....	4.54	4.88	5.08	5.53	5.95	3.00	1.40	0.81	0.50	0.23	.....	43.55
17....	3.90	3.80	4.16	4.35	4.90	5.10	2.80	1.35	0.75	0.46	0.22	40.48

proportioned throughout the period of run-off, and expressed as percentage of the total flow. The peak of such a flood is not created by the coincidental arrival of waters from all parts of the basin, but by the arrival of a flood wave, which has received its impetus and is developed and built up by direct run-off and lateral stream flow contributing throughout the combined length of the principal channels. Consistent similarity in shape and dimension of distribution graphs for a particular basin would indicate that the phenomenon occurs in practically the same manner for all unit-time storm periods. This suggests that a grouping of certain area units may be assumed, which, because of their position relative to the collection system, can be conceived as being enveloped by a time contour representing a period at the end of which the group has contributed its collective effect to the creation and acceleration of the flood wave. Under this assumption the distribution graph can be applied

to the area of the basin, giving its "area distribution graph," in acres (Column (6), Table 14). The conversion of the distribution graph (percentage of volume of flow) to the area distribution graph (percentage of area) in no way affects the computed results; that is, the computed results are the same as if there had been no resort to this assumption.

The application of the area distribution graph to rainfall excess (in inches per hour or in cubic feet per second per acre) distributes each minute of such rainfall excess throughout the period of run-off, in cubic feet per second. The horizontal accumulation of these 1-min increments of run-off produces the total run-off from the area at any minute, measured at the inlet (Column (24), Table 14).

As the rainfall is considered to be uniform over the entire area of 40 acres, and as each inlet area is contributing to the flow of the sewer, the flow, as it combines, will determine capacity. This synchronization is accomplished by determining the time taken by the run-off waters, as they combine, in passing through the various stretches of the sewer. A particularly satisfactory expression for average velocity in terms of the factors determinable in such a problem is that presented by R. L. Gregory and C. E. Arnold,[13] Associate Members, Am. Soc. C. E.

TABLE 15.—METHOD OF COMBINING FLOW FROM INLETS NOS. 1 AND 2 WITH THAT OF INLET NO. 3

Elapsed time, in minutes	Average Minute Discharge, in Cubic Feet per Second			Velocity		Time, in Minutes			Coincident discharge from Inlet No. 3, in cubic feet per second	Average Minute Discharge, in Cubic Feet per Second				
	From Inlet No. 2	From Inlet No. 1	From Inlets Nos. 1 and 2	In feet per second	In feet per minute	In transit from Inlet No. 2 to Inlet No. 3 (1,000 ft.)	From beginning of rainfall	Of arrival at Inlet No. 3		Inlet No. 4	Inlet No. 3	Inlet No. 2	Inlet No. 1	Below Inlet No. 4
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1	0.33	....	0.33	0.88	77	13.0	1	14.0	4.17	0.33	....	....	....	0.33
2	1.04	....	1.04	1.80	108	9.3	2	11.3	8.10	1.04	....	....	....	1.04
3	2.18	....	2.18	2.18	131	7.6	3	10.6	14.97	2.18	....	....	....	2.18
4	4.17	....	4.17	2.56	154	6.5	4	10.5	21.77	4.17	0.33	....	....	4.50
5	8.10	0.33	8.43	3.07	184	5.4	5	10.4	28.34	8.10	1.04	....	....	9.14
6	14.97	1.04	16.01	3.61	217	4.6	6	10.6	34.83	14.97	2.18	....	....	17.15
7	21.77	2.18	23.95	3.97	238	4.2	7	11.7	40.97	21.77	4.17	0.33	....	26.27
8	28.34	4.17	32.51	4.29	258	3.9	8	11.9	46.21	28.34	8.10	1.04	....	37.48
9	34.83	8.10	42.93	4.60	276	3.6	9	12.6	49.04	34.83	14.97	2.18	....	51.98
10	40.97	14.97	55.94	4.91	295	3.4	10	13.4	49.35	40.97	21.77	4.17	....	66.91
11	46.21	21.77	67.98	5.16	311	3.2	11	14.2	48.26	46.21	28.34	8.10	0.33	82.98
12	49.04	28.34	77.38	5.34	321	3.1	12	15.1	46.23	49.04	34.83	14.97	1.04	99.88
13	49.35	34.83	84.18	5.45	327	3.1	13	16.1	43.55	49.35	40.97	21.77	2.18	114.27
14	48.26	40.97	89.23	5.54	333	3.0	14	17.0	40.48	48.26	46.21	28.34	4.17	126.98
15	46.23	46.21	92.44	5.59	336	3.0	15	18.0	37.47	46.23	49.04	34.83	8.10	138.20
16	43.55	49.04	92.59	5.59	336	3.0	16	19.0	35.54	43.55	49.35	40.97	14.97	148.84
17	40.48	49.35	89.83	5.52	331	3.0	17	20.0	31.66	40.48	48.26	46.21	21.77	156.72
18	37.47	48.26	85.73	5.45	327	3.1	18	21.1	28.91	37.47	46.23	49.04	28.34	161.08
19	35.54	46.23	81.77	5.41	325	3.1	19	22.1	26.47	35.54	43.55	49.35	34.83	163.27
20	31.66	43.55	75.21	5.27	316	3.2	20	23.2	24.37	31.66	40.48	48.26	40.97	161.37
21	28.91	40.48	69.39	5.20	312	3.2	21	24.2	22.58	28.91	37.47	46.23	46.21	158.82

[13] "Run-Off—Rational Run-Off Formulas," *Transactions*, Am. Soc. C. E., Vol. 96 (1932), Equation (50), p. 1094.

The sewer system in the problem begins with Inlet No. 1, Inlets Nos. 2, 3, and 4 contributing at 1 000-ft intervals progressing down the sewer. The purpose of Table 15 is to show how the flow from Inlets Nos. 1 and 2 may be combined with that of Inlet No. 3, the flow from Inlets Nos. 1 and 2 (Columns (2) and (3)) having been combined in the same manner. Column (4) gives the combined flow from Inlets Nos. 1 and 2 as such flow enters the stretch between Inlet No. 2 and Inlet No. 3. The average velocity, in cubic feet per second, computed by Equation (2), is shown in Column (5) and is converted to velocity, in feet per minute, in Column (6), from which the time in transit through the stretch is determined and listed in Column (7). Column (8) gives the period from the beginning of the rainfall to that minute at which the increment of flow enters the stretch, which, added to the time in transit through the stretch (Column (7)), determines the time of arrival at Inlet No. 3.

In the meantime Inlet No. 3 has been discharging into the sewer. For instance, at the end of 20 min this inlet will have passed its peak discharge of 49.35 cu ft per sec and will be discharging at the rate of 31.66 cu ft per sec, while Inlet No. 2 will be delivering 40.48 cu ft per sec and Inlet No. 1, 49.35 cu ft per sec to the same point in the sewer.

If, then, the hydrograph of flow from Inlet No. 3 is adjusted to this position relative to Inlets Nos. 2 and 1 (Column (10), Table 15), the total flow, minute by minute, through the stretch between Inlets Nos. 3 and 4 becomes available. Columns (11) to (15), Table 15, show the arrangement of computations for the synchronized flow from the four inlets entering the stretch below Inlet No. 4.

No attempt has been made to take into account the changes in flow volume within 1-min intervals throughout the period of flow adjustment, involving the first and last 5 or 6 min. The synchronization, therefore, is less accurate for the extremes of flow, but is in agreement to the minute for the period between the 12th and 30th minutes (not all included in the published table).

To continue on the premise that fixed time intervals divide the drainage area into zones of like contributing characteristics, the area distribution graphs of the four inlets can be combined as the foregoing analysis has shown their flows to combine. The complete computation, of which Table 14 is an example, showed that the flow from Inlet No. 4 enters the stretch in the first minute; that from Inlet No. 3 reaches the stretch 3 min later; that from Inlet No. 2 at the end of 7 min, or 3 min later than that of Inlet No. 3; and the flow from Inlet No. 1 is 4 min later than that of Inlet No. 2.

Table 16 gives the sub-area combinations for each minute throughout the duration of run-off, producing an area distribution graph for the entire 40 acres (see Column (6)), applicable to the point of design below Inlet No. 4. This area distribution graph is now applied to the rainfall excess produced by the 10-yr storm of the problem in the same manner as that suggested in determining the run-off at the various inlets. The distribution and accumulation

TABLE 16.—AREA-DISTRIBUTION GRAPHS (UNITS ARE IN ACRES)

Elapsed time, in minutes (1)	Inlet No. 4 (2)	Inlet No. 3 (3)	Inlet No. 2 (4)	Inlet No. 1 (5)	Total (6)	Elapsed time, in minutes (1)	Inlet No. 4 (2)	Inlet No. 3 (3)	Inlet No. 2 (4)	Inlet No. 1 (5)	Total (6)
1.......	0.10	....	....	....	0.10	14...	0.30	0.60	1.10	0.50	2.50
2.......	0.20	....	....	....	0.20	15...	0.20	0.50	0.90	1.00	2.60
3.......	0.30	....	....	....	0.30	16...	0.10	0.40	0.70	1.70	2.90
4.......	0.50	0.10	....	....	0.60	17...		0.30	0.60	1.40	2.30
5.......	1.00	0.20	....	....	1.20	18...	10.00	0.20	0.50	1.10	1.80
6.......	1.70	0.30	....	....	2.00	19...		0.10	0.40	0.90	1.40
7.......	1.40	0.50	0.10	....	2.00	20...			0.30	0.70	1.00
8.......	1.10	1.00	0.20	....	2.30	21...		10.00	0.20	0.60	0.80
9.......	0.90	1.70	0.30	....	2.90	22...			0.10	0.50	0.60
10.......	0.70	1.40	0.50	....	2.60	23...				0.40	0.40
11.......	0.60	1.10	1.00	0.10	2.80	24...			10.00	0.30	0.30
12.......	0.50	0.90	1.70	0.20	3.30	25...				0.20	0.20
13.......	0.40	0.70	1.40	0.30	2.80	26...				0.10	0.10
										10.00	40.00

are made in the same manner as for the inlet. The result is the total run-off from the area, minute by minute, as it reaches the point of design, and is found to be the same as those of Column (15), Table 15, any slight differences being due to the adjustment in position between the four area distribution graphs to the nearest minute.

The advantages of having at any point under design an area distribution graph are obvious. With it various rainfall frequencies, rainfall intensities, infiltration rates, and run-off coefficients can be compared in terms of sewer dimension and cost. Another advantage of the hydrograph of flow, made available by the unit hydrograph, over a maximum only, as determined by other methods, lies in the ability to evaluate the effect of converting the several maxima of the hydrograph from sewer capacity to temporary back-water areas at the inlets. Particularly where the conditions produce distribution graphs, area distribution graphs, and hydrographs of flow of relatively short base and high peak, may this advantage become a factor in economical design.

It is possible, through the unit hydrograph, to analyze and correct what may be misconceptions in the "rational" method, a method which has enjoyed a wide use in the field of storm-sewer design and to the development of which the authors have contributed extensively. The method has back of it a ground of rationality that is entirely lacking in many of the older empirical formulas. Briefly, its theory is that a particular average rainfall intensity (of a given frequency) becomes the critical one for a drainage area when it is sustained throughout a period equal to the time of concentration for the area. The time of concentration is usually defined as that time necessary for the run-off from the "remote" portion of the area to reach the point of concentration. A further explanation of the theory is that even though higher rates will be reached for shorter "concentration" periods, run-off from them will be less than that from the critical intensity, because of the dissipating effect of surface pondage, together with the assumption that the rain will have stopped or so decreased in intensity before the end of each concentration period that the areas immediately adjacent to the outlet will have had time to relieve themselves of

their run-off waters before the "remote" waters reach that point. Likewise, rainfalls of longer duration than the concentration period will be less effective, having lower average rates.

If the average rainfall intensities given by Equation (2), which are applicable to St. Louis, Mo., are distributed with the distribution graph reduced from the authors' unit graph for Area C, and each duration period is considered an independent rainfall, or synthetic storm, it will be found that, regardless of the coefficient or deductive factor used to reduce the theoretical 100% run-off to the actual run-off, average rates of rainfall for shorter periods than the period established by the base of the distribution graph, will produce potential run-off greater than that produced by the critical intensity.

The hypothetical problem used in the foregoing demonstration of the unit hydrograph method will here be solved by the rational method. The "inlet time" will first be taken as equal to the base of the distribution graph for the upper inlet, which is 16 min. By acceptable methods, the time in transit through the proposed sewer is found to be 11 min, giving a time of concentration of 27 min. From the rainfall-intensity formula, Equation (2), the corresponding critical intensity of 3.64 in. per hr is computed. The coefficient, C, is $\frac{3.14}{3.64} = 0.86$, and the rational equation becomes, $Q \doteq C\ i\ A = 0.86 \times 3.64 \times 40 = 125.70$ cu ft per sec, producing a maximum flow which is 23% less than that developed by the unit hydrograph (Column (15), Table 15). Under the prevailing conception of concentration time, then, the rational method may give results that are too low, but the practical significance of this is lost in the wide range of values which have been given to the coefficient, C.

The foregoing conclusion, however, is predicated on the definition of concentration time as being equal to the base of the unit hydrograph of Inlet No. 1 plus time in transit, thus assuming that the "remote" area of the upper inlet is the last to contribute run-off to the point of concentration. The writer has found that, apparently, the unit hydrograph and rational methods may be brought into accord by defining concentration time as being equal to t_1, the time interval between the center of mass of rainfall and the flood peak or the "'lag" interval, or the time position of the maximum ordinate of the unit hydrograph, plus any time in transit involved in reaching the point of concentration, and by converting average rainfall rates into average rate throughout any minute, arranged in critical order.

Following are the results of applying the rational method, so modified, to the Inlet No. 1 area, as shown in Table 14: Concentration time $t_1 = 6$ min (Column (5)); average of first 6 min of rainfall = 5.42 in. per hr (Column (2)); coefficient, $\frac{4.92}{5.42} = 0.91$; and (see Column (24) Table 14); $Q = C\ i\ A = 0.91 \times 5.42 \times 10 = 49.25$ cu. ft per sec.

The concentration time for the point of design below Inlet No. 4 is determined by adding to the 'inlet time," or concentration time for Inlet No. 1, the time in transit through the sewer. The maximum discharge, as determined by the rational method, is as follows: The inlet time = 6 min; time in

transit = 11 min(see Table 15); concentration time = 17 min; average rainfall intensity for 17 min = 4.55 in. per hr; coefficient, $\frac{40.3}{4.55} = 0.89$; and, $Q = 0.89 \times 4.55 \times 40 = 162.20$ cu ft per sec (see Column (15) Table 15).

As it has been customary to estimate inlet time at 5 to 20 min, results from the rational method have probably not been greatly affected by the apparent inconsistency revealed in the foregoing comparison; rather has the analysis upheld the rational method as a quick and reliable means of estimating maximum flow in storm sewers.

LEROY K. SHERMAN,[14] M. AM. SOC. C. E. (by letter).[14a]—An important analysis of the rainfall-run-off relation on urban areas is presented in this paper, which is a step beyond the usual procedure based on the assumption of a uniform rate of rainfall during the storm period. Recently, the writer examined the rainfall and stream-flow records, due to forty storms, on a combined sewer area of 4 000 acres, in the Rock Creek Basin of the District of Columbia. The records were from self-recording gauges. In all these storm hydrographs the effect of varying intensity of rainfall produced a pronounced peak, in spite of the fact that storm durations generally exceeded the concentration period. This indicates that the assumption of a uniform rate of rainfall is not in accord with Nature's procedure and that, for certain areas, it may not be satisfactory. By an application of the unit hydrograph, the authors give a procedure which utilizes the actual varying intensity of the rainfall.

In forecasting run-off from rainfall data by the unit hydrograph (or by any other method or formula), the chief problem is to determine the factor expressing loss in run-off due to infiltration or soil capacity. With few exceptions, Fig. 15 indicates a close agreement between observed and computed rates of run-off. The authors used the factor of average percentage of run-off applied to the 100% run-off graph.

Fig. 16 suggested the application of the storage equation to develop or apply the factor of infiltration loss. The storage equation is:

$$\text{Inflow Volume} = \text{Outflow Volume} \pm \text{Storage} \qquad (7)$$

which may be expressed:

$$\text{Rainfall} = \text{Run-Off} + \text{Infiltration} \pm \text{Storage} \qquad (8)$$

There are two unknown quantities in Equation (8). However, channel-storage depth is related to run-off rates and, likewise, storage depth on a plane surface is related to run-off rates. For the latter case the storage depth, d, is the same as the hydraulic radius, R. Therefore, the discharge rate is, $Q = C \sqrt{S\, d} \times d$, and,

$$\frac{Q}{Q_1} = \frac{d^{\frac{3}{2}}}{d^{\frac{3}{2}}_1} \qquad (9)$$

in which Q_1 = the 100% run-off rate; d_1 = the storage depth corresponding to Q; Q = any given run-off rate; and d = storage depth corresponding to Q.

[14] Cons. Engr., Chicago, Ill.

[14a] Received by the Secretary March 30, 1935.

With the flow, Q_1, there is no infiltration loss. Therefore, Equation (7) can be applied to the data in Fig. 16 and d_1 may be determined for any rate of Q_1. The ordinate between the lines for accumulated rain and accumulated 100% run-off gives the depth of storage, d_1, at any time from the beginning of the storm. Values of d can now be computed by Equation (9). The values of d can be plotted for any given time in Fig. 16. These values will be on a line above, and somewhat parallel to, the line of accumulated 100% run-off. The ordinates between this line for d and the line for accumulated actual run-off give, by Equation (8), the accumulated loss to any given time.

The writer has applied the foregoing procedure for determining rate of loss per hour, or infiltration capacity, to the examples in Fig. 15. He obtained results as good, but no better, than the authors except in the case of the peak on Area *A* (Fig. 15). The writer also applied this storage equation procedure to a hypothetical case wherein he assumed the rates of rainfall and loss, in inches, as shown in Table 17. For comparison, the computed losses are also shown. Evidently, the procedure indicated by Table 17 did not offer any improvement over the use of the authors' application of average percentage of run-off.

TABLE 17.—HYPOTHETICAL CASE; COMPARISON OF TRUE VERSUS COMPUTED LOSSES

Day	Rain, depth	Loss, depth	Computed loss, depth
1	1.4	0.4	0.29
2	2.4	0.4	0.45
3	3.4	0.4	0.45
4	1.4	0.4	0.50
	......	1.60	1.69

Recently, the writer received a paper[15] by Robert E. Horton, M. Am. Soc. C. E., who makes successful use of the storage equation in the rainfall-run-off relation by taking cognizance of factors which the writer has neglected in this discussion.

To the present time the writer has secured the best results with the unit hydrograph method by applying the percentage of run-off directly to rainfall without the use of the intervening 100%-run-off graph or Bernard's pluviograph.[16]

This paper represents considerable careful work. The authors have presented valuable data and study applicable to problems in storm-sewer design and to the science of hydrology. Parts of the complete paper, as filed in Engineering Societies Library, New York, N. Y., are fully as important as the subject-matter submitted in the printed part. The writer refers particularly to the author's experiments on rates of infiltration during continuous rains.

[15] "Surface Runoff Phenomena," by Robert E. Horton, *Publication 101*, Horton Hydrological Laboratory, February 1, 1935, Edmonds Bros., Inc., Ann Arbor, Mich., Publisher.

[16] "An Approach to Determinate Stream Flow," by Merrill M. Bernard, M. Am. Soc. C. E., *Proceedings*, Am. Soc. C. E., January, 1934, p. 3.

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DISCUSSIONS

THE SILT PROBLEM

Discussion

By Messrs. Morrough P. O'Brien, Harry F. Blaney, W. W. Waggoner, and Philip R. R. Bisschop

Morrough P. O'Brien,[28] Assoc. M. Am. Soc. C. E. (by letter).[28a]—The compilation on the silting of reservoirs which is contained in this paper shows clearly the economic importance of the silt problem and the necessity for a continuation of investigations in this field. A paper by Dr. Fritz Orth[29] dealing with the same subject and published almost simultaneously contains much additional information including the data given in Table 10. All the numerical values have been converted to the same units as those used by the author.

The author mentions several estimates of the percentage of the total silt transportation which moves as bed-load. Any such estimate must be made on the basis of some arbitrary quantitative definition of what is meant by bed-load. As the tractive force moving the material increases, the region of bed transportation gradually expands until the concentration at or near the surface becomes appreciable and a condition of suspension is said to exist. Measurements show, and the theory of turbulent flow indicates, that the concentration of material should increase downward, the rate of increase becoming greater for larger materials. When materials in suspension are being transported over a mobile bed containing the same material, it is doubtful whether the bed has a well-defined surface along which material could be said to move as bed-load. In the laboratory, bed-load is defined as that material caught in a properly designed trap, whereas in studies of sedimentation in reservoirs or lakes it is the difference between the volume of material deposited and that accounted for by measurements of suspended sediment and discharge. Whatever the definition may be, the percentage of bed-load rises abruptly to

Note.—The paper by J. C. Stevens, M. Am. Soc. C. E., was published in October, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: February, 1935, by Harry G. Nickle, Jun. Am. Soc. C. E.; and March, 1935, by Messrs. E. W Lane, and Frank E. Bonner.

[28] Associate Prof., Mech. Eng., Univ. of California, Berkeley, Calif.

[28a] Received by the Secretary February 13, 1935.

[29] "Der Verlandung von Staubecken," von Fritz Orth, *Die Bautechnik*, June 19, 1934, Vol. 12, No. 26.

TABLE 10.—SILTING OF RESERVOIRS REPORTED BY F. ORTH

Reservoir	Stream	Drainage area, in square miles	Mean annual supply, in thousands of acre-feet	ORIGINAL CAPACITY		Period, in years	SILT DEPOSITED		
				Thousands of acre-feet	Percentage of annual supply		Total, in acre-feet	Percentage of original capacity	Annually, in acre-feet
Faal	Drau	5 140.	....	......	......	6.5	2 420	....	373.
Jettenbach	Inn	4 730.	9 090.	....	......	6	1 750	....	292.
(Genfer See)	Rhone	2 860.	....	72 100.	......	.....	......	....	2 400.
Pernegg	Mur	2 420.	....	4.0	......	1.5	284	70.0	203.
(Bodensee)	Rhine	2 360.	776.	....	......	20	8 010	....	401.
St. Denis du Sig	Sig	1 350	....	2.8	......	8	608	17.4	76.
Cheurfas	Meckerra	1 160.	23.3	10.1	41.7	50	4 950	48.8	99.
Gokak	....	1 080.	....	20.8	......	33	4 910	29.1	149.
Avignonnet	Drac	772.	890.	0.8	0.09	8	810	100.	101.
Quinson	Verdun	695.	....	1.1	......	5	730	67.9	146.
Medina Lake	Medina River	602.	....	253.	......	13	2 680	1.1	207.
Rosshaupten	Lech	550.	1 590.	....	......	7	1 780	....	255.
(Bieler See)	Aare	532.	1 710.	1 010.	58.6	20 16	5 430 2 030	0.5 0.2	272. 126.
Kallnach	Aare	525.	....	1.5	......	6	810	55.6	138.
Perolles	Sarine	487.	....	0.8	......	14	810	100.	57.6
(Thuner See)	Kander	414.	994.	5 270.	531.	152	46 000	0.87	303.
(Chiem See)	Tiroler Ache	392.	1 090.	1 790.	164.	34 1	2 630	0.15	68.9 160.
Saalach	Saalach	386.	1 010.	2.8	0.28	17	2 340	82.2	138.
Avisia	Avisia	369.	....	1.6	......	8	1 620	100.	203.
Djidovia	....	328.	....	1.6	......	.....	......	....	203.
(Vierwaldstadter See)	Reuss	321.	608.	9 560.	1 570.	27	3 200	....	118.
(Bodensee)	Bergrenzer Ache	321.	776.	39 400.	5 070.	24	2 510	....	105.
Loch Raven	Gunpowder River	306.	300.	1.6	0.52	20	1 330	85.0	66.5
Pont du Loup	Drac	290.	....	2.4	......	1.1	1 220	50.0	1 110.
Wallensee	Linth	240.	....	2 030.	......	51	3 030	0.15	60.1
Steyrdurchbruch	Steyr	222.	510.	6.7	0.12	22.5	575	84.	25.6
Cismon	Cismon	192.	446.	10.	0.64	10	1 420	14.2	142.
Monte Reale	Celina	168.	567.	1.	......	1	681	....	681.
Urfttalsperre	Urft	146.	146.	36.9	25.3	16	16	....	1.0
Wetzmann	Gail	125.	303.	0.5	0.16	1	486	100.	486.
Marklissa	Queis	120.	138.	12.2	8.82	25	152	1.25	6.1
Goldenstraum		108.		8.5	6.17	9	62	0.73	7.0
Pont	Armonoon	106.	373.	4.3	11.5	50	61	1.41	1.2
Lake Wichita	Holliday Creek	69.	....	14.0	......	25	490	3.48	19.5
Breitenhain	Weistritz	56.	51.1	6.5	12.7	15	112	1.72	7.5
Tlétat	....	50.	....	0.6	......	.....	......	....	17.8
Dambal	....	43.	....	2.6	......	48	462	17.7	9.6
Muchkundi	....	26.	....	16.2	......	41	2 590	16.0	63.2
Tarcento	Torre	24.	90.9	0.12	0.13	12.5	122	100.	9.7
(Wolfgansee)	Zinkenbach	22.	52.4	501.	957.	18	108	....	6.0
Grosbois	LaBronne	11.	8.9	7.5	83.7	150	24	0.33	0.16
Lete	Lete	11.	....	0.8	......	17	11	1.4	0.66
Laragnina	Gorzento	9.8	....	0.8	......	20	324	38.1	16.2
Marinkop	....	6.3	....	1.0	......	46	169	17.0	3.6
Pontebba	Vogelbach	3.9	7.8	....	......	18	227	....	12.6
Brux	Einsiedlerbach	3.2	2.8	1.3	47.1	.....	......	....	0.8
Tillet	Tillet	2.1	0.73	0.42	57.8	.....	24	5.77	0.02
Saifnitz	Luscharibach	1.7	3.39	0.02	0.7	1	24	100.	24.3
McKinney	....	1.4	....	0.01	......	10	12	100.	1.2
Camperdown	Umlassfluss	.....	....	1.86	......	16 8	1 000 316	53.5 17.0	62.4 39.7
Holtwood	Susquehenna	.....	28 400.	54.8	0.2	18	13 000	23.7	721.
Bhatodi	Mehekari	.....	6.77	3.50	51.7	50	2 610	74.6	51.9

100 as movement starts and then decreases as material is thrown far enough into the stream to be classed as suspended material. Therefore, it is different for every stage of the river.

Some progress has been made in the application of the theory of turbulent flow to the distribution of suspended sediment[30] and the greatest obstacle at present is the lack of precise data. W. Schmidt[31] has developed the basic equation and some data are available for checking it, but much more will be needed before its validity can be fully established. The basic data needed are: Measurements of the vertical distributions of velocity and sediment; and the fall-velocity of the material in suspension. Useful supplementary data are: The size of the bed material; the hydraulic radius or depth; and the energy slope. This need is mentioned in the hope that engineers taking sediment samples will also obtain the other data mentioned.

In Table 10 the variation in silt deposited per thousand parts of water supply, by volume, is from 0.028 to 16.3. Interesting as these data are, they do not provide a means of predicting the life of a proposed reservoir and this is the principal problem involved. Most of the factors causing this wide variation are mentioned by the author. Further progress in this field appears to require a breaking down of the problem into two phases, of which one is the volume of sediment transported by the stream and the other, the characteristics of reservoirs as a silt trap. Laboratory and field studies can contribute much valuable information on these points, but any prediction of the rate at which sediment will be transported to a reservoir is necessarily subject to all the uncertainty involved in forecasting stream flow.

On the assumption that a given project will continue to need the volume of effective storage originally provided for it, the monetary damage caused by silting should not be computed from the cost of the original reservoir but from the cost of the necessary additions. As the sites which are less expensive per unit of storage are developed first, silting involves an increasing rate of damage as it proceeds and it may ultimately be impossible to provide sufficient storage capacity at any cost. At present, construction of additional reservoirs appears to be the practical solution, but the time may come when by-pass channels and other silt-controlling works, costing perhaps more than the dam itself, will be found to be economically justified. Such structures are now used successfully at diversion dams, and it seems possible that conditions at some storage dams might warrant the construction of works for diverting a portion of the silt.

HARRY F. BLANEY,[32] ASSOC. M. AM. SOC. C. E. (by letter).[32a]—The author calls attention to facts which have often been overlooked in the past. The paper is a valuable contribution to the literature on the subject of silt. Since the writer is more familiar with conditions along the Lower Colorado

[30] "Review of the Theory of Turbulent Flow and Its Relation to Sediment Transportation," by Morrough P. O'Brien, Assoc. M. Am. Soc. C. E., Am. Geophysical Union, Section of Hydrology, 14th Annual Rept., p. 487 (1933).

[31] "Die Massenaustauch," von W. Schmidt, H. Grand, Hamburg, Germany, 1925.

[32] Irrig. Engr., Bureau of Agri. Eng., U. S. Dept. of Agriculture, Los Angeles, Calif.

[32a] Received by the Secretary March 7, 1935.

River and the Middle Rio Grande, this discussion will be confined to the areas served by these streams through the reaches mentioned.

As indicated by Mr. Stevens, silt is a menace to irrigated agriculture in some sections. This problem is one of the most serious confronting the people of New Mexico to-day, especially in the Rio Grande Valley. This valley supports about one-third of the people of the State, as well as a large population in Texas. The silt problem of the Middle Rio Grande may be divided into three phases: (*a*) The silting of the river channel above Elephant Butte Reservoir; (*b*) the deposition of silt in the Elephant Butte Reservoir; and (*c*) the accumulation of coarser silt in the main stream channel below the Elephant Butte Dam, due to lack of seasonal flood flushing. The Federal Governments of the United States and Mexico have started a project to rectify the conditions below the dam. Recently, the State Planning Board of New Mexico has recommended that a relief project be undertaken to control erosion on the Rio Puerco, which is the principal source of silt in the Upper Rio Grande water-shed, and reports[33] that:

"The Rio Grande Valley is in great danger of destruction as a habitation for man, due to the rapid erosion of its tributaries and silting of the river itself. In approximately 50 years the great Elephant Butte Dam and irrigation works connected with it will lose their usefulness, due to the silting up of the reservoir and the destruction of its storage capacity."

It is estimated in this report that silt endangers an investment of $100 000 000 in land and improvements. The river above the Elephant Butte Dam for a distance of 150 miles has such a moderate fall that in many places a large quantity of silt is deposited. This building up of the river channel is increasing flood hazards and water-logging the bordering lands. Surveys made in 1934 show a rise of the river channel at La Joya (just below the mouth of the Rio Puerco) of 7 ft since 1918; at Albuquerque, 2 to 4 ft; and at Alameda, 1 to 3 ft. The most important stream that enters the Rio Grande above Elephant Butte Reservoir is the Rio Puerco, and its control would go a long way toward solving the silt problem.

As indicated by the author, the Boulder Reservoir will not completely solve the silt problem on the Lower Colorado River. On March 1, 1935, about 110 000 acre-ft of water had accumulated in Boulder Reservoir, and the water leaving it was clear. However, an additional silt load will undoubtedly be picked up by the river below the dam for some time to come. If the experience on the Rio Grande below Elephant Butte Dam is repeated on the Colorado River, the clear water released from either Boulder Reservoir or from Parker Reservoir will pick up a load of silt and scour the bed, progressively lessening in depth but extending in distance. With an average discharge of 15 000 to 20 000 cu ft per sec, the bed of the river will probably be washed in time so that most of the fine silts will be removed and only the bed sands left to line the channel. Occasional floods entering the river below the Boulder Dam or the Parker Dam will carry large quantities of silt and will cause a temporary increase of suspended silt in the river at the Imperial Valley Diversion.

[33] "Silt Control on the Rio Puerco," by S. R. DeBoer, Mimeographed Rept., New Mexico State Planning Board, November, 1934.

The character of the silt will be changed materially, since it will be a mixture of silt picked up from the bed of the river. This will consist of silts from different tributaries of the river, no one of which will predominate.

There is a difference of opinion among engineers as to how many years it will take the regulated river to clean its channel of the finer silts and cease to carry any appreciable quantity of material in suspension.

With the river unregulated, a low percentage of suspended silt usually occurs in the early summer at the peak of the annual flood resulting from the melting of the snows in the upper water-sheds, while the river water generally carries the highest percentages of silt in the late summer months or fall when erratic floods are caused by rains in areas drained by its lower tributaries.

The writer believes that the low percentage of silt carried in suspension by the river in past years is a good index of what may be expected immediately after Boulder Reservoir becomes effective in desilting the water. The annual quantity of silt carried in suspension in the river will be reduced by at least one-half at the Imperial Valley Intake, and for the most part, the large quantities of objectionable fine silt which have been present in the water during the peak of the irrigation season will be eliminated.

When the new Imperial Diversion dam and heading are completed about five miles above Laguna, a portion of the present main canal serving the Yuma Project will probably be abandoned, and its water brought through the upper end of the All-American Canal. Laguna Dam will remain, to serve as a drop structure in the river, controlling recession of grade below Imperial Dam. It has been estimated that it will take from three to five years to complete this project. Meanwhile, Imperial Valley will receive water through the present system, and silt conditions will be about the same as heretofore.

The effect of silt on the cost of operation and maintenance of the irrigation system in Imperial Valley has been a serious one. The use of silty water for irrigation as it comes from the river means the handling of an enormous quantity of silt, which, of necessity, must be deposited in the various canals of the system, or on the land itself. In some sections it is necessary to keep dredgers continually at work in order to maintain the canals and laterals in condition to carry the required quantity of water. The disposal of this silt dredged from the canals is also fast becoming a perplexing problem. The banks of the canals are constantly being raised and widened. The Imperial District spends approximately \$500 000 per yr for silt disposal. In addition, there is the cost of cleaning farm ditches and re-leveling of land because of silting. A survey by the U. S. Bureau of Agricultural Engineering in 1933 indicated that the total cost of cleaning ditches and re-leveling land as a result of silt deposits in Imperial Valley ranged from 55 cents to \$4.63 per acre per yr and averaged \$1.80. This is in fair harmony with the usual estimate of \$2.00. Applying the latter figure to the 400 000 acres under irrigation in the Valley, the cost would reach \$800 000 per yr. Adding to this the amount spent by the Imperial Irrigation District gives a total annual cost of silt disposal of more than \$1 250 000.

Several different plans for the exclusion of silt from the All-American Canal, both bed and suspended load, have been considered. Research investigations being conducted by the U. S. Bureau of Reclamation and the Bureau of Agricultural Engineering are expected to furnish the fundamental data for the preparation of the final designs for desilting works at the new Imperial Dam. The Bureau of Reclamation has been successful in eliminating about 50% of the suspended silt and most of the bed silt in the desilting operations at Laguna Dam.[34] The writer feels confident that when the new Imperial Heading is finally built it may be expected to eliminate practically all the bed silt and at least 50% of the suspended silt carried by the river water. With the regulated river carrying 50% of its former quantity of suspended silt only 25% of the former suspended load would enter the irrigation system. However, it will be advisable to retain some silt in the water so as to prevent the growth of moss. Under these conditions the annual expenses to the individual farmer of cleaning farm ditches and re-leveling land, and to the Imperial Irrigation District for silt disposal, would be reduced considerably.

W. W. WAGGONER,[35] M. AM. SOC. C. E. (by letter).[35a]—In an exhaustive and comprehensive manner Mr. Stevens has presented "the silt problem" for discussion. His description of the silting presents a dark and important picture. The tabulation of the quantities of silt transported by the various rivers is indeed startling. It shows how much greater the mud flow is from the sedimentary and fragmentary formations than from the crystalline rocks.

The answer to the question of preventing the silting of a reservoir is to build another one above it, to impound the silt, and also to encourage the deposition of silt above the reservoir as in the case of Lake McMillan which the author mentions. It would be interesting and constructive to the discussion of the paper if a detailed description of the silt impounded at Lake McMillan could be given.

The gold mines of California are drained by the Yuba River, a tributary of the Sacramento River. This area has been the scene of the most important operations in hydraulic mining. Due to his long contact with the gold-mining industry, the writer has made a detailed study of the débris problem in its various phases and desires to record the débris deposit in the Yuba River, and its relation to the subject under discussion.

Denudation is more than the eroding of the mountains, and the filling of the rivers with sand, or the forming of deltas at the river's mouth. It is one of the grandest features of the Creator's plan for maintaining the fertility of the soil for the benefit of mankind throughout the ages. Consider the nature of the soil: Professor E. W. Hilgard[36] has stated that the soil is

[34] "Silt in the Colorado River and Its Relation to Irrigation," by the late Samuel Fortier, M. Am. Soc. C. E., and Harry F. Blaney, Assoc. M. Am. Soc. C. E., *Technical Bulletin* 67, U. S. Dept. of Agriculture, 1928, p. 58.

[35] Hydr. and Min. Engr., Nevada City, Calif.

[35a] Received by the Secretary March 18, 1935.

[36] "Geochemistry," by E. W. Hilgard, *Bulletin 491*, U. S. Geological Survey.

composed of 3 to 5% of organic matter, and 97 to 95% of inorganic matter from which the plants obtain the nine mineral elements that enter into their growth, namely, phosphorus, potassium, calcium, magnesium, sodium, iron, silicon, chlorin, and sulfur. These elements combine to form the common minerals such as feldspar, mica, and hornblende, which compose the crystalline rocks.

It may be stated that the valleys are the granaries from which the people obtain the major portion of their food, and that the mountains are the storehouses from which the depleted soils receive a new supply of fertilizer. The action of the elements is to break up the cementing power that holds the minerals in a solid mass, and to form individual grains that compose the fine, inorganic part of the soil. These, in turn, are also oxidized, and the elements are slowly dissolved, and carried into the soil for plant food. It is remarkable how a thin soil, with an adequate precipitation can produce a great forest growth, due to the abundance of plant food.

For example, consider Central California, on the western slope of the Sierras, with its great forests and its abundance of precipitation. During the dry seasons the erosion is negligible, but during the wet, or flood, seasons it becomes of great volume. The storms beat against the mountain slopes where they dissolve the plant food and carry it away with the particles of sand into the ravines and creeks. These waters concentrate into streams which debouch into the valley. By so doing, great canyons are formed and mountains leveled. With the heavy grades, and high velocities, the streams are capable of carrying all the silt brought to them. When the valley has reached the flatter grades, sand and gravel are deposited upon their beds. The finer material is carried onward. During floods the water spreads over the valley; some of it soaks into the ground where, at least, a part of the dissolved plant food is left behind to act as a fertilizer.

With the occupation of the valley these flooded lands are very attractive for farming. Levees are constructed to protect these lands from the floods and to get rid of the so-called nuisance. The finer sands, together with the dissolved matter, of necessity are transported to the bays and the ocean.

As an illustration of the fertility of such land may be cited the case of a farm with 270 acres in wheat, 10 acres of which were measured and shown to produce 50 470 lb. In 1858, the wheat from this tract was so rich in gluten that it was in great demand in the markets of New York, N. Y., and Liverpool, England. To-day, 1 000 lb per acre is an average yield, but the wheat is so poor in gluten that it has to be mixed with other wheat from new lands to make a marketable flour, due to the depletion of new plant food.

Consider next the history of a great débris deposit upon the Yuba River, which is comparable to that at Lake McMillan. After a long period of industrial depression in the United States, gold was discovered in far-away, unknown California, on January 24, 1848. This discovery electrified the world. A man could go to California, accumulate a fortune by gathering gold from its sands, and then return home to live in peace and comfort for the remainder of his days!

Never was there such an army of active, energetic, educated, and resourceful men as gathered in California. While many followed their original plan, most of them remained to found a State. They had to be housed and fed. This led to the opening of farms, and the setting out of orchards. All the best lands were brought into cultivation. Ten years after the discovery of gold there was a greater acreage of farms on the Yuba drainage area than there is to-day. These industries needed lumber to carry on their operations and for the towns that grew up not only in the mountains but in the valley as well. In 1859, there were ninety-nine saw-mills in the three counties covering the Yuba drainage area. The long, dry seasons were ideal for lumbering. It can well be imagined how the ground was plowed up by the ox teams that were used for snaking the logs to the mills. This continued for about thirty years, or until the wonderful forests (which would be hard to find elsewhere) were exhausted.

In November, 1861, a cycle of wet winters began, which continued for twenty years, and produced the worst floods in the later history of the State. During December, 1861, and January, 1862, four great cyclonic disturbances occurred that enveloped the entire Pacific Coast, and produced floods, from the Columbia River to and including Los Angeles, Calif. In the Sierras it was commonly called "The Deluge." Upon the middle belt of the Sierras during these months 75 in. of rain fell, and near the summit 42 in. of rain and 50 ft of snow fell; 5 and 6 in. per day fell during a 4-day period. A seasonal total of 109 in. occurred in the gold belt.

Combining the stories of the historians of that period the water flowed into the valley at a greater rate than could be discharged through the Carquinez Straits, and the Golden Gate. From the foothills of Mt. Shasta to those leading up to the Tehon Pass the great California Valley was a vast lake not unlike the shape and size of Lake Michigan. The tides ceased to act, and a stream of thick, muddy water flowed through the Golden Gate, which discolored the ocean for a distance of 40 miles from the land. The bays became fresh, killing the oysters which were planted near Oakland.

The Yuba River discharged its quota coming from its various tributaries surcharged with mud and sand gathered from the mines, fields, and forests. The Lower Yuba River bottoms was a narrow valley, 1 mile to 3 miles wide, carved in the peneplain of the valley. The mud and sand buried the farms and orchards 15 ft deep; with the succeeding years came more floods and with them the deposit increased in depth until it was raised about 10 ft above the general valley plain. Levees were constructed to confine the deposit. One of the results that occurred as soon as the deposit reached the level of the valley floor was to raise the water-table on the adjacent lands. It was brought out in the litigation which followed that they were known as "redlands" and were poor even for grazing. After the raising of the water-table they were used for general farming.

In the early days the gold was recovered from the gravel by washing in sluice-boxes, the gravel being moved principally by picks and shovels. In 1870, mechanical equipment was invented by which means a considerable

volume of water could be used under high heads, and a large volume of gravel could then be moved at a reasonable cost. These giant machines are now used in the construction of hydraulic-fill dams and similar structures. This equipment led to the consolidation of small holdings and the formation of large companies financed in San Francisco, New York, and London. Reservoirs, the highest in the United States, were built, and long lines of canals were constructed to the mines. These structures were completed about 1876.

At the same time the agitation against the débris flow that began after the floods of 1862 and (were charged against the mines) took the form of litigation which culminated in the injunction that closed the mines in 1884. During the litigation, the State Engineering Department of California was created. Elaborate surveys of the navigable rivers and the Yuba River were made. In 1894, the Army engineers made a series of borings to determine the depth of the deposit. These borings were supplemented by those of the gold dredgers.

Combining the various surveys it was found that the deposit which was 20 miles long, had a maximum width of 3 miles, covering 16 000 acres and containing 600 000 000 cu yd. It was 20 ft deep at the river's mouth, 35 ft deep at the edge of the foothills, and 80 ft deep 5 miles higher up the Yuba River. The grade of the original bed was 5 ft per mile. After the fill was made, the grade per mile was $2\frac{1}{2}$ ft at the mouth, 10 ft at the middle zone, and 20 ft on the upper reaches.

The deposit soon became covered with a dense growth of willows. It remained intact with a few, low-water channels coursing through it. About 25 yr ago the Government enlarged the main low-water channel until now (1935) it will carry a moderately large flood. Much of the adjacent land that was supposed to be destroyed forever was cleared and planted to pears, the uncleared land being held at more than $250 per acre.

In making the deposit the law of transporting material is probably the same as that which obtains in the sluices of the hydraulic mines. It is known that with water flowing 10 to 12 in. deep the transporting capacity varies directly as the slope. Furthermore, with water flowing 2 ft deep, more material will be carried per cubic foot per second than with water flowing 10 in. deep. Mathematically, the transporting capacity of force, F, varies as V^2.

It is interesting to record that fields which were denuded of their soil are now covered with second growth pine and brush. One variety of brush known as "sweet birch" is especially valuable as a cattle food. One enthusiastic cattle man claimed that it was nearly as good as alfalfa, and made it a practice to gather and scatter the seed over his range; while another farmer claimed that a good soil could be formed by it in 25 yr.

The lesson of the débris deposit on the Yuba River is of far-reaching application. By spreading the water over a wide area by low barriers, and by encouraging brush growth, the silt will be deposited and can be raised to any height. The material thus impounded will be prevented from moving down stream, filling reservoirs, or from entering the navigable rivers where it would

have to be dredged at a cost of 6 or 7 cents per cu yd. Below the barriers the stream will then be able to scour its bed and increase its capacity. The cost of such impounding will be less than 1 cent per cu yd. It is conservation in the highest degree.

PHILIP R. R. BISSCHOP,[37] ASSOC. M. AM. SOC. C. E. (by letter).[37a]—In South Africa the silt problem is no less severe than that encountered in the southwestern part of the United States. Of late years particularly, the rapid rate of silting recorded in many of the irrigation reservoirs has caused considerable concern. In a number of cases the very existence of productive centers of population is threatened, and for what period of time it will be possible to preserve the present water supply is a moot point. Indeed, it is the definite policy of the present Government to limit the extensive further program of irrigation development to those rivers that are least subject to silt encroachment.

Typical of some of the conditions encountered in this country is the case of the Sundays River, in the Cape Province. Rising at elevations of 6 000 to 7 000 ft, this river drains about 6 000 sq miles on the south escarpment of the Sneeuwberg Range which divides the Great Karroo Plateau of 4 000 to 5 000 ft elevation from the Small Karroo Plateau of 1 500 to 2 000 ft. In the foothills of this inclined plane, on the main branch of the river, is situated the Van Ryneveld's Pass Reservoir with a catchment of 1 500 sq miles. At the southern edge of the Small Karroo Plateau, where the river starts breaking through the coastal ranges, is situated Lake Mentz Reservoir with a catchment area of 4 500 sq miles.

The entire drainage area can be divided roughly into the "Mountain" Section and the "Flats" Section, the first-named reservoir taking on part of the mountain topography whereas the second takes care of the remainder of the mountain area and all of the "Flats" Section.

The average annual rainfall varies from 14 to 18 in. in the mountains to from 10 to 14 in. in the flats. Approximately 80% of this rain is concentrated in the summer months, particularly February and March. Consequently, the river flows are essentially ephemeral in character and are liable to exceedingly high flood peaks in very short periods of time, falling again with only slightly less suddenness.

Previous to the building of the reservoirs, intermittent irrigation was practised extensively, by what is known as "flood furrow irrigation"; that is, individual small systems diverting water into fairly large canals in order to obtain a maximum of water in a minimum of time whenever available. Such irrigation was necessarily of an exceedingly up-and-down character, incapable of sustaining much more than "snatch" crops and occasional cuttings of alfalfa. With the completion of the reservoirs, land development and settlement occurred along lines quite similar to those experienced in the United States. To-day, both reservoirs sustain intensively developed

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[37a] Received by the Secretary April 3, 1935.

irrigated areas, subdivided in small holdings on which a struggling population is obtaining a living from citrus, deciduous fruits, alfalfa, and cereal crops.

Below the reservoirs, therefore, there is a cropped area totally dependent on conserved water, and above them, a pastoral area subject to severe droughts and high floods. In the past, the latter area has been subjected to extensive over-grazing and although to-day the more progressive farmers are fully alive to the decreasing grazing capacity of their farms and are actively endeavoring to rehabilitate their veldt, the damage already done by soil erosion is such that it has become a national problem against which the incumbent Government has organized an intensive campaign of research, subsidization of anti-erosion work and rehabilitation of the original vegetation. The benefits of such work can necessarily only become apparent over a considerable period of time and cannot be other than partial in overcoming all erosion. In the meantime, the lower lying reservoirs are gathering the accumulated burden. For example, Lake Mentz, completed in 1922, has silted up to 42.3% of its original capacity (see Table 11), say, in twelve years of life, and Van Ryneveld's Pass is estimated to have lost 16.4% of its capacity between 1925 and July 1, 1933.

TABLE 11.—DEPOSITION OF SILT IN THE LAKE MENTZ STORAGE DAM (ORIGINAL STORAGE CAPACITY, 94 000 ACRE-FEET)

PERIOD BETWEEN SURVEYS;		CAPACITY, IN ACRE-FEET		PERCENTAGE LOSS OF STORAGE	
From:	To:	Inflow	Silt deposited	By volume; Column (4) / Column (3) × 100	Cumulative total
(1)	(2)	(3)	(4)	(5)	(6)
November 30, 1922*	April 30, 1924	95 250	2 200	2.3	2.3†
May 1, 1924	March 30, 1926	111 770	2 900	2.6	5.2‡
April 1, 1926	November 30, 1927	24 630	620	2.5	6.0‡
December 1, 1927	March 31, 1929	443 910	10 000	2.25	16.6‡
April 1, 1929	June 30, 1930	175 070	4 030	2.3	20.8‡
July 1, 1930	December 27, 1931	116 140	2 670	2.3	23.7§
December 28, 1931	January 4, 1932	440 000	9 320	2.1	33.6§
January 5, 1932	June 30, 1933	131 840	3 030	2.3	36.75§
July 1, 1933	December 31, 1934	227 440	5 230	2.3	42.3§

* Date of closure. † Based on actual survey. ‡ Mean of Column (5). § Based on previous survey findings.

In the case of Lake Mentz, the raising of the dam has already begun and the levels of other reservoirs (such as Lake Arthur on a tributary of the adjoining Great Fish River) will have to be raised in the near future. Such periodic raising of the dam walls is necessarily limited, if not by physical features, certainly by financial considerations. Already, in a manner very parallel to that experienced in the United States, the administrators of many of the South African irrigation projects have found it impossible to meet their redemption and interest rates, and the Government, which financed the projects, has found it necessary and expedient to write off approximately $25 000 000—the entire capital loans and arrear interest payments of a

large number of projects. In agreeing to these "write-offs," the Government stipulated an annual assessment of approximately $0.30 per acre for the special purpose of establishing in each project a reserve fund to cover the cost of periodical heightening of the dam walls. The amount of this assessment cannot possibly be expected to cover the increasing cost of preserving the water supply. What then is to become of the endangered projects? Where further reservoir sites are available, or where the existing reservoir sites are not as yet fully developed, the problem resolves itself to an extension of the already widely held viewpoint that irrigation development is essentially a national function and that, as such, the Government should not only finance and subsidize—either directly or indirectly—the original cost, but also that of preserving the supply.

Further supplies are not always available, however, and existing reservoir sites must ultimately become fully developed if they are not developed already. Furthermore, public opinion may not support the conception that it is the function of the Government to preserve a water supply for an unlimited period. There appears to be only one solution, namely, to recognize frankly that some irrigation projects are doomed to a definite span of life and will have to revert back to the original type and manner of flood irrigation, to prepare the irrigators against such time and contingency, and, in the meantime, to delay the end as long as possible by actively combating sheet and river erosion within each catchment area.

The detailed technical data pertaining to the deposition of silt in the Van Ryneveld's Pass Dam are:

Drainage area, in square miles	1 475
Mean (estimated) annual supply, in acre-feet	38 000
Original full supply capacity, is acre feet	64 200
Percentage of annual supply, $\left(\frac{64.2}{38.0} \times 100 =\right)$	169
Date of reservoir completion	1925
Elapsed time, in years, between completion of reservoir and last capacity survey	6
Water supply, in acre-feet, during the six-year period	134 600
Silt Deposited, in Acre-Feet:	
Total	5 244
Annual	874
Percentage of original capacity	7.4
Percentage of original capacity, by volume $\left(\frac{5\ 244}{134\ 600} \times 100 =\right)$	3.9

Since the date of the last capacity survey (February, 1931), a flood of 100 000 acre-ft passed through this reservoir, on January 1 and 2, 1932. By July 1, 1933, it was estimated that, in all, 10 500 acre-ft of silt had been deposited with a reduction in capacity of 16.4 per cent.

Lake Mentz is a non-overflow, gravity dam provided with five Stoney gates, 30 ft wide and 25 in. high. In March, 1928, a flood of 356 000 acre-ft entered the reservoir, of which 310 000 acre-ft passed through the gates. During the flood of January 1 and 2, 1932, furthermore, 421 000 acre-ft were by-passed through the gates. At Van Ryneveld's Pass, the dam is of the overflow, gravity section, type and has only been in action once when 46 000 acre-ft were spilled.

It will be noticed from Table 11 that the percentage, by volume, of silt retained at Lake Mentz is only 2.3 as against that of 3.9 at Van Ryneveld's Pass. This may be accounted for, possibly, by the lesser run-off and lesser erosion from the flatter and, at the same time, lower rainfall portion of the catchment area as against the higher run-off and greater erosion in the Mountain Section. It may also be accounted for by the probability that a certain quantity of the suspended and bed load is scoured out by the gates as against the skimming action of an overflow type of dam. The two reservoirs also provide an interesting illustration of the effect of the size of the mean annual run-off and the storage capacity on the life of the reservoir. Although Lake Mentz has a lower volume percentage of silt and a 50% larger storage capacity, the annual run-off from the larger catchment is such that its life, theoretically, is 30 yr as against 50 yr for Van Ryneveld's Pass.

Considerable work has been done in the United States in estimating the life of reservoirs by silt sampling. In this connection, the writer would like to recall the work and method of approach adopted by Mr. C. H. Warren, while he was Circle Engineer of the South African Irrigation Department. Silt samples were taken in bottles and analyzed both for specific gravity and silt percentage by weight, and a curve was obtained expressing the relationship. From samples taken from the bed of the Great Fish River, Mr. Warren obtained a weight of 52.8 lb per cu ft of dry silt, or 84.6% of the weight of 1 cu ft of water. A hundred cubic feet of water, containing 1% by weight of silt, therefore, will deposit $\frac{100}{84.6}$, or 1.18 cu ft of silt. As the silt percentage

TABLE 12.—Ratio of Percentage by Volume to Percentage by Weight for Increasing Loads of Silt

Specific gravity (1)	Weight of 1 cubic foot, in pounds (2)	Percentage of silt by weight (3)	Ratio, $R = \frac{\text{Percentage by volume}}{\text{Percentage by weight}}$ (4)	Percentage of silt by volume (5)	Specific gravity (1)	Weight of 1 cubic foot, in pounds (2)	Percentage of silt by weight (3)	Ratio, $R = \frac{\text{Percentage by volume}}{\text{Percentage by weight}}$ (4)	Percentage of silt by volume (5)
1.000	62.42	0.00	1.182	0.00	1.040	64.92	5.86	1.230	7.21
1.005	62.74	0.73	1.188	0.87	1.050	65.55	7.32	1.241	9.09
1.010	63.05	1.46	1.194	1.74	1.060	66.17	8.79	1.253	11.03
1.015	63.36	2.20	1.200	2.64	1.070	66.79	10.25	1.265	12.97
1.020	63.67	2.93	1.206	3.53	1.080	67.42	11.72	1.277	14.97
1.025	63.99	3.66	1.212	4.44	1.090	68.04	13.18	1.289	16.99
1.030	64.30	4.40	1.218	5.36	1.100	68.67	14.65	1.301	19.05

increases, so also will the weight per cubic foot of water laden with that silt increase. Therefore, he calculated a ratio, R, expressing the relationship between the percentage by volume and the percentage by weight for increasing loads of silt in the water (see Table 12).

The factor, R, is obtained as follows: Assume a quantity of water loaded with 4.40% of silt by weight. From Table 12, 1 cu ft of water thus laden will weigh 64.30 lb and, hence, 100 cu ft will deposit 64.30×4.40, or 282.92 lb., or $\frac{282.92}{52.8}$, or 5.358 cu ft, and $R = \frac{5.358}{440} = 1.218$.

With the foregoing information, Mr. Warren traced the silt load of the Tarka River, discharging into Lake Arthur Reservoir. Over a 3-yr period an average of 2.76% by weight, or an equivalent of 3.31% by volume, was found.

From the published records of the United States Reclamation Service, Mr. Warren assumed that the rolling bed load was 25% of the suspended load, thus making a total silt load of 4.14% by volume. A silt survey of the reservoir at the end of the 3-yr period showed a silt content of 4.34% by volume, a difference of less than 5 per cent. Although the foregoing result may be fortuitous in some small degree, it constitutes a logical attempt, nevertheless, to take into account the varying degrees of suspended silt carried at different stages during the same flood as well as the varying silt load in different floods resulting from different rates of intensities of rainfall.

The writer has recently had an opportunity of viewing the anti-erosion work conducted both in South Africa and in the United States, and has formed the opinion that this work, judiciously planned and co-ordinated to local circumstances and supplemented by further work in the river channels proper, can reduce the silting of reservoirs materially.

Although it is axiomatic, of course, that a certain degree of erosion will occur, and that silt will always be present, due to ever-present dynamic geological processes, these factors, nevertheless, have been considerably augmented by the careless and haphazard processes of mankind and, in particular, by his agricultural activities. With the opening of new territories, for example, areas have been put under the plow that should never have been plowed. To-day, washes and gullies (or "dongas," as they are known in South Africa), are in evidence, that have widened from 50 to 300 ft "within the living memory of man." The writer has often asked himself the question whether, if the white man had not entered upon his activties in these areas, these selfsame gullies and "dongas" would not still to-day be nearer to the 50-ft width instead of the 300-ft width.

Obviously, it will be impossibe both physically and economically to overcome entirely that portion of erosion which can be attributed to Man's activities. A large part, however, can be overcome. The results obtained at Lake McMillan and Zuni Reservoirs, discussed by Mr. Steven, can be considered at least as definite and established indications of what can be accomplished. In the former case, the accidental propagation of tamarasks provided an effective silt trap. In the latter, anti-erosion work of a limited character

was conducted on only a portion of the catchment area. In neither of these cases can it be said that the catchments were subjected to a systematic, judicious, and complete program of anti-erosion work, yet in both instances the rate of silting during the past 10 to 15 yr has been reduced to less than 40% of that experienced formerly.

It was with great interest, therefore, that the writer acquainted himself with the principles and aims of the United States Soil Erosion Service and the manner and methods in which these are executed and adapted to each local circumstance. In his opinion, they deserve the serious attention of the profession.

With the careful study that soil erosion is receiving on each of the Soil Erosion Service projects, chiefly from the more agricultural aspects, and the judicious execution of methods to combat these phases, benefits on the resulting silting must be inevitable. If these are next followed up by the more purely engineering aspect of combating erosion in the water channels proper —such as the maintenance of river alignments, the prevention of under-cutting the banks, the formation of vegetative silt traps, the construction of débris basin, as well as the further work of fire protection, the writer does not feel that he is over-optimistic in anticipating a reduction in the silt percentage by volume on many streams to 50% of that occurring at present.

Where engineering works can thus profit, it behooves the profession to pay sincere and serious attention to, and offer constructive co-operation in, the planning and execution of such work. Both in the United States and in South Africa, the writer feels, there is considerable scope for the Civil Engineer to participate beneficially in the soil erosion aspect of the silt problem.

The writer wishes gratefully to acknowledge the data supplied him by Mr. A. S. Bridgman, Maintenance Engineer of the Sundays River Irrigation Board, in regard to Lake Mentz Reservoir.

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DISCUSSIONS

ANALYSIS OF MULTIPLE ARCHES

Discussion

By Messrs. L. E. Grinter, N. M. Newmark, T. Y. Lin, A. H. Finlay, and A. W. Fischer

L. E. Grinter,[5] Assoc. M. Am. Soc. C. E. (by letter).[5a]—The second exposition in *Proceedings,* of the application of the Cross method of moment distribution to the analysis of multiple arches, is contained in this paper. The first was a discussion[6] of Professor Cross' paper on the "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Donald E. Larson, Jun. Am. Soc. C. E. Mr. Hrennikoff's presentation is highly satisfactory and should be read widely by those who have become familiar with the Cross method.

The extension of the conception of balancing moments and shears to the complex problem of the analysis of a series of arches on elastic piers has intrigued the interest of numerous investigators. The writer made his own extension in 1927 and has taught the theory of multiple-arch analysis by the general method of successive corrections[7] to his graduate students since 1929. It is interesting that the general method of successive corrections, devised to apply to any frame in which joint translation occurs, becomes essentially the same as the method described by Mr. Hrennikoff when it is applied to a multiple-arch system.

It is perhaps unfortunate that the author does not generalize his method of attack and explain its wide application, for the reader is more than likely to gain the impression that the paper deals with a group of special devices, of use only in the analysis of multiple arches. The simple explanation that follows makes the method generally applicable to all multiple-bay or multiple-story structures:

1.—With joints fixed against both rotation and translation, determine the fixed-end moments and the reactions at the ends of each loaded span.

2.—Balance moments with joints held against translation and determine the changes in the restraining forces that exist at the joints.

Note.—The paper by Alexander Hrennikoff, Esq., was published in December, 1934, *Proceedings.* This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

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[5a] Received by the Secretary, January 10, 1935.

[6] *Transactions,* Am. Soc. C. E., Vol. 96 (1932), p. 127.

[7] *Loc. cit.,* p. 18; and Vol. 98 (1934), p. 611.

3.—Eliminate all artificial joint forces by permitting the joints to translate without rotation under the action of an equal and opposite set of forces. Determine the new set of fixed-end moments introduced by this translation.

4.—Repeat Steps 2 and 3 as many times as necessary (seldom more than twice) to reduce the unbalanced moments and artificial joint forces to negligibly small factors.

The importance of placing emphasis upon these simple steps is that they attach a physical significance to the procedure which is so likely to be obscured when use is made of a table, such as Fig. 11. The method of successive corrections can be arranged into a mechanical procedure, but it is doubtful whether any important advantage is obtained. Of course, an orderly arrangement of calculations will reduce the labor of analysis and the probability of error; but over-standardization will make any procedure so highly mechanical that the various steps lose their physical significance and the method is then likely to become unintelligible to most readers. The primary reason why the Cross method has become widely popular is that each step of the procedure has a physical significance, and there is no reason why the general method of successive corrections should not retain this important characteristic.

The only difficulty experienced by the writer's students in the use of this method for continuous arch analysis has been in regard to the signs of the thrusts at the tops of the piers. This difficulty is largely eliminated by the use of arrows to indicate the direction of thrust, instead of the conventional signs. The suggestion is in line with the desire to maintain the greatest possible physical significance for each step of the process. The convention that a positive bending moment at the end of a member tends to rotate the adjacent joint clockwise has been found most satisfactory.

The detailed procedure by which the constants for moment or thrust distribution are obtained, is of no great importance. Mr. Hrennikoff seems to prefer formularization by means of the neutral point conception, whereas the writer would commonly choose the simple direct application of the column analogy.[8] A knowledge of the column analogy for obtaining distribution constants and a full understanding of the four listed steps of successive corrections make totally unnecessary the remembrance of any special terminology or special devices for the analysis of continuous arch systems.

The determination of influence lines for multiple arches is reasonably simple if the shape of the deflected load line corresponding to the proper unit displacement is found by the aid of the conjugate-beam theory. In this case it is particularly convenient to produce the proper unit distortions, successively, by the use of the column analogy. There is no particular reason for choosing between the determination of influence lines for arch forces or pier reactions. In either case one must obtain three key influence lines for each span after which these may be combined by statics to obtain any influence line desired.

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[8] "The Column Analogy," by Hardy Cross, M. Am. Soc. C. E., *Bulletin No. 215*, Univ. of Illinois, Eng. Experiment Station, Urbana, Ill.

The writer is hopeful that Mr. Hrennikoff's paper may help to convince designers that the proper analysis and design of a system of continuous arches is no longer an impossibly difficult task, or, for that matter, not even a particularly tedious task. Certainly, there is no real justification for neglecting the elastic properties of a pier when such neglect might conceivably endanger the structure. Much "water has passed under the bridge" in the last few years in regard to the theory of structures, and this paper is clearly indicative of the changes that are taking place.

N. M. NEWMARK,[9] JUN. AM. SOC. C. E. (by letter).[9a]—A method of analyzing multiple arches is presented in this paper which Mr. Hrennikoff states "is based on the well-known principle of moment distribution." A discussion of similar methods that have appeared in previous American literature may be of some interest.

It appears that the author's analysis of multiple-span arches is based upon a solution for the movements of the pier tops due to a given loading. These movements are then converted to forces and moments in the spans and piers. The method is similar, therefore, to the slope deflection method. Equations may be written for the forces (moments, thrusts, and shears) at the end of a member, either arched or straight, in terms of the displacements of that end, and of the far end, of the member.[10] Summing the forces at a joint (for all the members meeting at that point) in terms of the displacements and equating this sum to the unbalanced force at that joint yields a group of relations between the unbalanced forces and the joint displacements necessary to balance these forces. These are the equations used by Mr. Hrennikoff. When only two arch spans and a pier are considered (the far ends of the arch spans being fixed and vertical deflection of the pier being neglected), there are only two degrees of freedom of movement of the joint; hence, there are two unknowns and two equations to solve. When the displacements are found, the forces may be determined directly. When there are more spans, each equation will contain more unknowns, but the work of solution may be performed in the manner indicated by the author.

The general case has been treated by Mr. A. P. Hjort[11] who solves the equations by successive elimination of the unknowns, which is rather a tedious process when the number of unknowns is large; but for some types of structure, the coefficients of the unknowns are such that an accurate solution is possible without unwarranted precision in intermediate computations.

Professor Cross has suggested that these equations may be solved by successive convergence or successive approximations.[12] In an unpublished thesis submitted to the University of Illinois in partial fulfillment of the

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[9a] Received by the Secretary February 4, 1935.

[10] See, for example, "The Column Analogy," by Hardy Cross, M. Am. Soc. C. E., *Bulletin 215*, p. 72, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

[11] "Design of Continuous Arches on Elastic Piers," by A. P. Hjort, *Proceedings*, Am. Concrete Inst., Vol. XXIX, 1933, p. 143.

[12] "Statically Indeterminate Structures," by Hardy Cross, pp. 115–118, 1926.

requirements for the degree of Master of Science in 1930, D. E. Larson, Jun. Am. Soc. C. E., solved the equations in the manner suggested by Professor Cross.

Certain tricks facilitate the solution of the equations, and the author's paper demonstrates some of these tricks.

The concept of moment distribution and force distribution lends itself readily to the solution of this problem. The distribution of a moment at a given joint amounts to allowing that joint to rotate until the sum of the moments in the members meeting at the joint is equal to the unbalanced moment that existed before rotation. Thus, in the two-span, single-pier, arch system the unbalanced moment and the horizontal force at the pier top, due to any loading on the arches or piers, can be distributed by a rotation and a translation of the pier top. It will be found, however, that in general such a rotation balancing the moment, will introduce an unbalanced thrust at the pier top. Furthermore, a horizontal movement of the top of the pier balancing the thrust, will introduce an unbalanced moment at that point.

However, the joints may be balanced successively until convergence in some particular cases where convergence does obtain; for example, see the discussion by Mr. Larson[13] of Professor Cross' paper on moment distribution.

The foregoing explanation may be extended, of course, to a multiple-span structure and, in terms of the moment distribution concept, moments and thrusts may be distributed at all the joints and carried over; then at each joint the thrust due to the moment distribution and the moment due to the thrust distribution are added to the carry-over moments and thrusts to obtain the new unbalanced moments and thrusts at that joint. In a number of cases, however, the unbalanced moments and thrusts are almost as great as the original forces, and convergence may be very slow. It is possible to obtain more rapid convergence by use of a procedure such as that given by the author when a movement of the pier top is found to balance both moment and thrust at the given joint.

Mr. Hrennikoff does not use the procedure of carry-over and redistribution, but prefers to take into account the actual degree of fixation of the far end of the members. However, for the example which he gives, the convergence will be so rapid that it is scarcely necessary to go through the exact computations. Note that in the author's method two equations in two unknowns must be solved to distribute the forces at each joint.

Professor Cross has recommended[14] a moment-distribution procedure to be used for arches on elastic piers which eliminates the solution of simultaneous equations. If, instead of distributing moments and forces at the joint, these are distributed at a point away from the joint so located that a unit rotation of this point will produce no unbalanced thrust at the point and a unit translation of the point will produce no unbalanced moment at the point, then moments and forces can be distributed directly. It might be noted that this is equivalent to solving the two equations in two unknowns

[13] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 127.

[14] *Loc cit.*, pp. 152–154.

of Mr. Hrennikoff's method by substituting for them two equations which contain one unknown each and hence may be solved directly. Mr. Hrennikoff's equations might have been set up in terms of the "neutral point" with this simplification. Note that the neutral point will change in position if the far ends of the members are not considered fixed. Professor Cross proposes carrying over the changes in thrust along certain thrust lines in the different members to shorten the computations.

The essential difference, then, between Professor Cross' method[15] and that of Mr. Hrennikoff is the use of the neutral point and of successive convergence for solving the equations.

The rapid convergence of the Cross procedure is surprising. The writer has had occasion to use it in the analysis of a number of three-span arches on elastic piers. In this problem, when the structure happened to be symmetrical, considerable simplification was possible; by using the scheme of symmetry and anti-symmetry no carry-overs were necessary and all distributions were made directly.

It is gratifying to note the distinction the author makes between analysis and design. So far as design is concerned refinements in analysis are not warranted. It is almost impossible to determine accurately the stresses in a multiple-span structure, because of the variation in elastic properties of the different spans and between different parts of each span. Furthermore, the results are always complicated by the effect of the superstructure. The elastic constants for arch spans may be modified as much as 100%, or even more, by the effect of the deck.

It is worse than futile to compute influence lines for a multiple-span arch neglecting the effect of the deck. It is possible, of course, to compute such influence lines for total moment, thrust, or shear at a section through both rib and deck when, by some means or other, fixed-end influence lines and elastic constants for the composite structure are obtained.

T. Y. Lin,[16] Jun. Am. Soc. C. E. (by letter).[16a]—The principle of moment distribution has been applied to the analysis of multiple arches in many ways, but the procedure suggested herein is among the very simplest.

It would probably be better to use the "kip-inch" unit instead of the "kip-foot" unit for the fixed-end moments. Not only will this eliminate much confusion and misunderstanding, but it will enable the application of the reciprocal theorem[17] to check many of the calculated values. For example, in Fig. 11, Items 1 and 2, in the column headed "B_A", applying that theorem, it is known that h due to $\alpha = \frac{1}{E}$ must equal m due to $\Delta = \frac{1}{E}$, and, therefore, $-0.574 = -4.78 \times 12 \div 100$. The same is true in the column headed "B_B", and the column headed "Joint B".

[15] For formulas for the location of the neutral point and for the distribution and carry-over factors, as well as a complete description of the moment-distribution concept applied to multiple-arch analysis, see "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E., pp. 316–330, 1932.

[16] With Ministry of Railways, Chinese National Govt., Nanking, China.

[16a] Received by the Secretary February 9, 1935.

[17] "Elastic Arch Bridges", by McCullough and Thayer, pp. 271 and 306.

For Items 9 and 10, in the column headed "C_B": $-0.463 = -3.85 \times 12 \div 100$. Furthermore, in the column headed "Joint C": $16.26 = 135.6 \div 100$. For Items 40 and 41, in the column headed "Conditions", Δ due to $m = 1$ must equal α due to $h = 1$ and, therefore, $0.819 \div 12 = 0.0683$. Evidently, when more joints are considered, more such checks will be available.

A. H. Finlay,[18] Assoc. M. Am. Soc. C. E. (by letter).[18a]—The problem of analyzing multiple arches on elastic piers is one in which structural engineers are evidencing an increasing interest and Mr. Hrennikoff's paper, presenting as it does a novel and very direct method of analysis, is happily timed. The author's method presents a nice blending of two points of view. Starting with the basic idea of distribution factors and fixed-ended force functions he finds, in a simple and original manner, the joint movements necessary for equilibrium and, from these movements, the force functions to be combined with the original ones in order to get the final values.

Professor Cross has presented[19] a most ingenious solution of this problem, using his principle of force distribution in its most general form throughout. A description of this process will be of assistance in appraising the value of Mr. Hrennikoff's solution. For convenience of reference, the structure, symbolism, and sign conventions in this discussion are the same as those chosen by Mr. Hrennikoff.

If unbalanced force functions (that is, thrust and moment) are distributed in the usual manner many cycles will be necessary before a suitable balance is obtained. This is due, in part, to the fact that when, say, Joint B (Fig. 12),

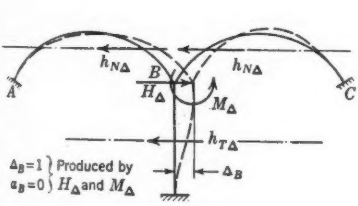


Fig. 12.

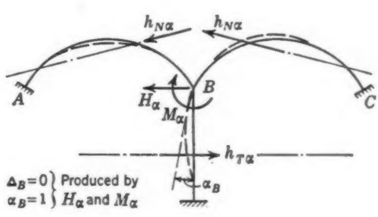


Fig. 13.

with adjacent joints locked, is translated horizontally without permitting rotation (in order to distribute horizontal thrusts), the thrusts ($h_{N\Delta}$ and $h_{T\Delta}$) produced in each adjoining arch and the pier by the moment, will produce in general, an unbalanced moment at Joint B. In other words during the translation the joint will have to be held against rotation. Similarly, when the joint is rotated without permitting translation (as when distributing moments) the horizontal components ($h_{N\alpha}$ and $h_{T\alpha}$) of the thrusts produced in each adjoining arch and the pier (Fig. 13) by the movement will produce, in general, an unbalanced thrust at Joint B. In other words, during the rotation of the joint it will have to be held against translation.

[18] Asst. Prof. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[18a] Received by the Secretary March 2, 1935.

[19] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 152 *et seq.*; and "Continuous Frames of Reinforced Concrete," by Messrs. Cross and Morgan, p. 316 *et seq.*

The lines of action and the magnitudes of the forces shown in Figs. 12 and 13 may be found in many ways. For convenience of comparison with the author's paper his h_Δ-forces evidently act along the thrust lines of Fig. 12 and his h_α-forces evidently represent the horizontal components of the forces acting along the thrust lines in Fig. 13, each of which passes through the neutral point of its respective member. The thrust lines in Fig. 13 are best located perhaps by finding their vertical distances from Points A and B. To facilitate comparison these distances, in the author's terminology, are, respectively, $\frac{m_{Fa}}{h_{Fa}}$, $\frac{m_{Na}}{h_{Na}}$, for an arch, and $\frac{m_{Ta}}{h_{Ta}}$ for the pier.

The foregoing comments have been intended to emphasize two points: First, the nature of the forces set up in each member of the assembly by each movement of the joint; and, second, the annoying unbalancing of one set of force functions caused by distributing the other set, which is inevitably associated with any distribution of force functions as long as such distribution is effected by pure translation and pure rotation of the joints themselves. Such a method of distribution, of course, will eventually produce a solution, and an interesting example of such a solution has been presented by Donald E. Larson, Jun. Am. Soc. C. E.[20]

Professor Cross has discussed this difficulty of slow convergence and has suggested how, in large measure, it may be avoided. He distributes force functions by pure translation and pure rotation of a point so chosen that it translates only under the action of a suitably directed force applied to the point and rotates only under the action of a couple applied to the point. He well styles this point the neutral point of the joint. In all cases this is a point on the vertical through the joint if the pier be incompressible. It may be defined in general as that point in which the resultant of the centroidal forces in each member of the assembly, accompanying pure translation of the joint, intersects this vertical. Since a force, such as H_Δ (Fig. 14), applied

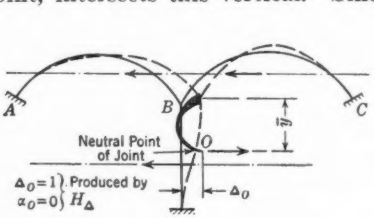


FIG. 14.

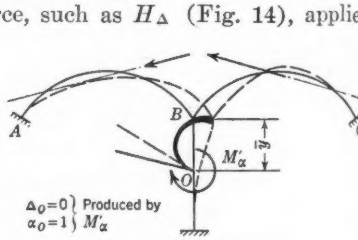


FIG. 15.

at this point causes only translation, the reciprocal theorem (of which that of Maxwell is a special case) shows that a couple, such as M'_α (Fig. 15), applied at the same point will cause it to rotate only.

It will be interesting to note the lines of action of the forces produced in each member of the assembly by pure rotation and pure translation of such a point. These lines are shown in Figs. 14 and 15. As before, the

[20] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 127.

magnitudes of the various forces can be found in different ways. One way is by considering the deflection of the neutral point of each member accompanying each specified movement of the point, O, and from these known deflections the forces accompanying them may be computed. For purposes of comparison it may be noted that the forces set up in the arches and pier in Fig. 14 are the same, of course, as those in Fig. 12 whereas, in Fig. 15, since the joint, B, has moved a distance, $\bar{y}$, to the right and has rotated through a unit angle, the forces set up are those in Fig. 13 plus $\bar{y}$ times those in Fig. 12. These relations are of no special interest, of course, except as they afford a convenient means, in this specific example, of checking numerical quantities used against those given by the author.

The distribution and carry-over factors may be found from the forces shown in Fig. 14 and Fig. 15, but it is more convenient to find them directly from consideration of the forces produced in each member of the assembly by a unit force and a unit couple, respectively, applied to the point, O. For the structure under consideration these forces are shown in Figs. 16 and 17.

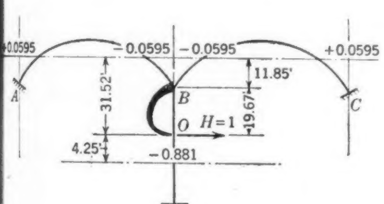


FIG. 16.

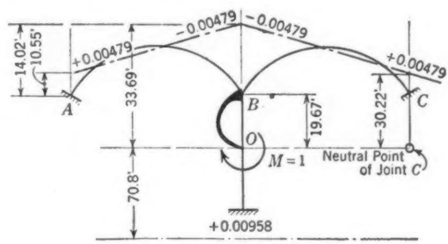


FIG. 17.

The magnitudes of the horizontal components of the thrusts are written on their lines of action. Since the sign conventions follow those of the author the forces considered are those which the members exert on their terminal points. Positive directions are to the right for thrust and clockwise for moment. In both diagrams, therefore, Arch A-B is pulling to the left on Point B and to the right on Point A.

Distribution factors at Point O for thrust are -5.95% for each arch and -88.1% for the pier. Carry-over factors for thrust are evidently equal to the foregoing values and, for equilibrium, of opposite sign. Distribution factors for moment at Point O are, evidently, -0.00479 (33.69) 100 $= -16.1\%$ for each arch and 0.00958 (-70.8) 100 $= -67.8\%$ for the pier. The carry-over factors for moment may be obtained in the same manner by multiplying the thrust by its appropriate lever arm, or they may be found from the distribution factors by multiplying by the appropriate ratio of distances. For example, the part of any unbalanced moment at Point O that is carried over to the neutral point of an adjacent joint, such as Point C, is $+16.1 \dfrac{(30.22)}{(33.69)} = +14.5$ per cent. The plus sign merely means that, in this example, the carried-over thrust due to rotation is exerting a clockwise

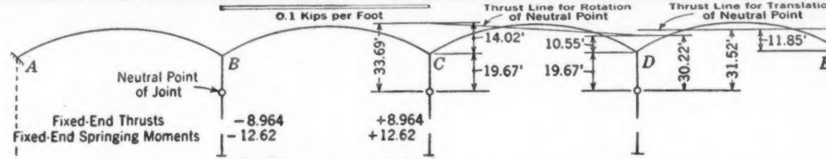
moment on the neutral point of Joint *C*. Summarized briefly, the foregoing means that unbalanced force functions at the neutral point of any joint are distributed among the members at such a joint with signs opposite to that of the unbalanced force function while they are carried over to the neutral points of adjacent joints with signs the same as that of the original unbalanced force function.

It should be realized clearly that, when distributing unbalanced thrusts by translating the neutral point, the distributed thrusts will cause no unbalanced moment at the neutral point. This follows, of course, from the definition of the neutral point, but a clear physical picture of this phenomenon can be obtained by realizing that the moments of the arch and pier thrusts in Fig. 14, or in Fig. 16 must balance about the neutral point of the joint. Similarly, when distributing moments by rotation of the neutral point, the distributed moments in the arches and pier will cause no translation of that point; that is, the horizontal components of the thrusts in the arches and pier accompanying such distribution must total zero as may be seen in Fig. 15, or in Fig. 17. This is the central idea in Professor Cross' ingenious method of distributing force functions. It follows from this that, during the actual distributing, no account need be kept of distributed functions but only of those parts of the unbalanced functions which are carried over to the neutral points of adjacent joints, since it is only these latter quantities that can unbalance any previously balanced neutral point.

The writer cannot emphasize too strongly the need of visualizing with the aid of Figs. 16 and 17 just what happens when the force functions are distributed. When thrusts are distributed, the distributed thrusts act, as has been stated, along the lines of action shown in Fig. 16. When moments are distributed the distributed moments set up thrusts which act along the lines of action shown in Fig. 17. If these two physical aspects are clearly understood the method of translating changes in thrusts into changes in moment, and *vice versa,* will be obvious.

Fig. 18 contains the full solution of all terminal forces for one load position. It should be noted that about one-half the tabular matter is devoted to the distribution proper, the remainder being given over to the tiresome but necessary translation of the various changes in thrust into final thrusts and moments. This latter is only simple arithmetic combined with the laws of statics, but, of course, it must be done. It is in this last step of the work that the distributed thrusts which, heretofore, have been ignored, enter the calculations. Since these distributed thrusts are equal numerically and opposite in sign to their respective carried-over thrusts they may be written by inspection. The end arches and the piers need not appear in the original distribution. This reduces the work materially but, of course, eliminates any check of the arithmetic by statics.

If many loading conditions are to be investigated the part of the work devoted to distribution can be reduced materially (as Professor Cross has demonstrated) by first distributing a unit horizontal force and a unit couple at each neutral point, beginning near the center of the structure. In this



Carry-over +5.95% of Thrust and +14.5% of Moment

All Thrusts, in Kips
All Moments, in Kip-Feet

Item No.		A	B (arch AB)	B (arch BC)	C (arch BC)	C (arch CD)	D (arch CD)	D (arch DE)	E
	Fixed-End Thrusts			−8.964	+8.964				
	Fixed-End Springing Moments			−12.62	+12.62				
1	Fixed-ended thrusts			−8.964	+8.964				
2	Distribute and carry-over thrusts. Record			+0.533	−0.533		+0.533		
3	Only carried-over thrusts			−0.032	+0.032	+0.032	−0.032		
4				+0.004	−0.002	−0.002	+0.004		
5	Sum of carried-over thrusts (Lines 2+3+4)			+0.505	−0.503	+0.030	+0.505		
6	Reduce to moments at neutral point (multiply by 31.52)			+ 15.9	− 15.9	+0.9	+15.9		
7	Fixed-ended moments at neutral point (±12.62±8.964 (19.67))			−188.0	+188.0				
8	Total unbalanced moment at neutral point			−172.1	+172.1	+0.9	+15.9		
9	Distribute and carry-over moments to neutral point			+25.0	−24.9	+2.3	+25.0		
10	Record only moments carried over			− 3.3	+ 3.6	+3.6	− 3.3		
11				+ 1.0	− 0.5	−0.5	+ 1.0		
12					+ 0.1	+0.1			
13	Sum of carried-over moments (Lines 9+10+11+12)			+22.7	−21.7	+5.5	+22.7		
14	Reduce to thrusts (divide by 30.22)			+0.750	−0.718	+0.182	+0.750		
15	Distribute and carry-over thrusts			−0.032	+0.045	+0.045	−0.032		
16				+0.005	−0.002	−0.002	+0.005		
17	Sum of carried-over thrusts (Lines 15+16)			−0.027	+0.043	+0.043	−0.027		
	Thrusts Caused by Translation of Neutral Points:								
18	(*a*) Due to carried-over thrusts (Lines 5+17)	−0.460	0	+0.478	−0.460	+0.073	+0.478	0	+0.073
19	(*b*) Due to distributed thrusts (not shown)	0	+0.460	+0.460	−0.478	−0.478	−0.073	−0.073	0
20	Total thrust caused by translation (Lines 18+19)	−0.460	+0.460	+0.938	−0.938	−0.405	+0.405	−0.073	+0.073
	Thrusts Caused by Rotation of Neutral Points:								
21	(*a*) Due to carried-over moments (Line 14)	−0.718	0	+0.750	−0.718	+0.182	+0.750	0	+0.182
22	(*b*) Due to distributed moments (not shown)	0	+0.718	+0.718	−0.750	−0.750	−0.182	−0.182	0
23	Total thrust caused by rotation (Lines 21+22)	−0.718	+0.718	+1.468	−1.468	−0.568	+0.508	−0.182	+0.182
24	Thrust due to translation and rotation (Lines 20+23)	−1.178	+1.178	−2.406	−2.406	−0.973	+0.973	−0.255	+0.255
25	Fixed-ended thrusts			−8.964	+8.964				
26	Actual thrusts in arches	−1.178	+1.178	−6.558	+6.558	−0.973	+0.973	−0.255	+0.255
	Moments of Springings Caused by:								
27	(*a*) Thrusts due to translation (Line 20)(11.85)	− 5.5	+ 5.5	+11.1	−11.1	− 4.8	+ 4.8	−0.9	+0.9
28	(*b*) Thrusts due to carried-over moments (Line 21) (10.55)	− 7.6	0	+ 7.9	− 7.6	+ 1.9	+ 7.9	0	+1.9
29	(*c*) Thrusts due to distributed moments (Line 22) (14.02)	0	+10.1	+10.1	−10.5	−10.5	− 2.5	−2.5	0
30	(*d*) Fixed-ended moments			−12.62	+12.62				
31	Actual springing moments (Lines 27+28+29+30)	−13.1	+15.6	+16.48	−16.58	−13.4	+10.2	−3.4	+2.8
32	Thrusts at pier tops (by subtraction)		+5.380		−5.585		−0.718		
33	Moments at pier tops (by subtraction)		−32.08		+29.98		−6.8		

FIG. 18:

way a set of distribution factors is built up which eventually allows of direct distribution at any neutral point. If this modification is used it will be seen that Mr. Hrennikoff's alternative method is similar to it in general idea.

The writer, familiar with both Mr. Hrennikoff's and Professor Cross' analysis of this problem, is inclined to favor the former if for no other reason than that it gives springing thrusts and moments directly without the arithmetic of Lines 24 to 33 in Fig. 18. On the other hand, with Mr. Hrennikoff's method there are the equations to be solved, although with his alternative method the number of these equations is greatly reduced. Both methods of course are powerful tools, each well adapted to the task, and the writer earnestly commends a study of both of them to those concerned with this type of analysis.

A. W. Fischer,[21] Esq. (by letter).[21a]—The author is to be commended for his efforts to simplify the solution of one of the widely discussed problems arising in structural engineering. His method is short and simple when the spans are all the same and symmetrical and the arch has been analyzed for fixed-end moments and thrusts; but when the fixed-end moments and thrusts are not given and the axis does not follow the axis as given in the papers[3] by Charles S. Whitney, M. Am. Soc. C. E., the method is still quite long especially for the analysis of three spans or more. In the latter case it seems that a simple algebraic solution developed by Mr. H. Yu[22] is at least as rapid, and the results will agree very closely since both methods are based on the deflection of curved beams.

When applied to the two-span arch, Example I (which is one of the cases treated by Mr. Whitney[3]), the results obtained by the author's method prac-

TABLE 4.—End Distribution Factors of Ribs and Pier for Numerical Example I (Divided by E)

Rotation Factors			Displacement Factors		
Equation No.	Factor	Value	Equation No.	Factor	Value
(7)	h_{Fa}	0.5736 in.²	(12)	$h_{F\Delta}$	0.004034 in.
(7)	h_{Na}	—0.5736 in.²	(11)	$h_{N\Delta}$	—0.004034 in.
(5)	m_{Na}	—10.11 in²-ft	(13)	$m_{N\Delta}$	—0.04780 in-ft.
(6)	m_{Fa}	4.815 in²-ft	(13)	$m_{F\Delta}$	0.04780 in-ft.
(16)	h_{Ta}	2.151 in²	(19)	$h_{T\Delta}$	—0.007493 in.
(17)	m_{Ta}	—76.93 in²-ft	(20)	$m_{T\Delta}$	0.1794 in-ft.

tically agree with those of Mr. Whitney. In order to demonstrate the simplicity and brevity of the author's method when the fixed-end moments and thrusts are given, and to show how the results compare with other methods,

[21] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[21a] Received by the Secretary, April 8, 1935.

[3] *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 931; and Vol. 90 (1927), p. 1094.

[22] "Reinforced Concrete Multiple-Arch Bridge on Elastic Piers," by H. Yu, 5a Si-Siao Shaun-Ma-Chuang, North end of Pei-Hsin-Hwa-Chieh, Peiping, China.

it seems desirable to present the calculations in detail for a unit vertical load at a definite point. (All computations in this discussion were made with a 20-in. slide-rule.)

The values in Table 4 have been calculated by using those given in Mr. Whitney's papers[3] and substituting in Equations (5) to (20), inclusive,

TABLE 5.—END MOMENTS AND THRUSTS, TWO-SPAN ARCH BRIDGE.
(Thrusts, h, are in pounds; and moments, m, are in foot-pounds)

Item No.	Conditions	End Moments and Thrusts for Member:									
		A_B	Joint B								C_B
			B_A		$B_{B'}$		B_C		Total, Joint B		
		m	h	m	h	m	h	m	h (Columns (3), (5), and (7))	(Columns (4), (6), and (8))	m
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	$a = \frac{1}{E}$	+4.815	−0.5736	−10.11	+2.151	−76.93	−0.5737	−10.11	+1.0038	−97.15	+4.815
2	$\Delta = \frac{100}{E}$	+4.780	−0.4034	−4.780	−0.7493	+17.94	−0.4034	−4.780	−1.5561	+ 8.38	+4.780
3	Fixed-ended forces at 0.9 point.	+6.38	+0.191	+1.52	0.0	0.0	0.0	0.0	+0.191	+1.52	0.0
4	$a = \frac{0.02778}{E}$	+0.13	−0.016	−0.28	+0.060	−2.14	−0.016	−0.28	......	......	+0.13
5	$\Delta = \frac{14.07}{E}$ ..	+0.67	−0.057	−0.67	−0.105	+2.52	−0.057	−0.67	......	......	+0.67
6	Two-span forces....	+7.18	+0.118	+0.57	−0.045	+0.38	−0.073	−0.95	0.0	0.0	+0.80

of this paper. Table 5 is arranged in a manner similar to Table 3 of the paper and gives the values for a unit vertical load at the 0.9th point[23] of the two-span arch on an elastic pier as given in Example I in Mr. Whitney's papers.[3] Moments are expressed in foot-pounds and thrusts in inch-pounds. (For other values of the two-span forces for unit vertical loads at Points 0.8 to 0.1, as calculated by the author's method, see Table 6.)

TABLE 6.—END MOMENTS, IN FOOT-POUNDS, AND THRUSTS, IN POUNDS, FOR UNIT VERTICAL LOADS AT THE TENTH POINTS BY THE AUTHOR'S METHOD.

Ratio of span, length $\frac{x}{l}$	Horizontal thrust, H_A	Bending Moment		Ratio of span, length $\frac{x}{l}$	Horizontal thrust, H_A	Bending Moment		Ratio of span, length $\frac{x}{l}$	Horizontal thrust, H_A	Bending Moment	
		M_A	M_B			M_A	M_B			M_A	M_B
(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
1.0	0.000	0.00	0.00	0.3	0.841	+3.51	−7.24	0.4	0.507	+5.79	+6.39
0.9	0.118	−7.18	+0.57	0.2	0.509	+2.91	−8.24	0.5	0.596	+6.72	+7.62
0.8	0.406	−9.41	+1.47	0.1	0.190	+1.39	−6.20	0.6	0.563	+6.32	+7.28
0.7	0.745	−7.97	+1.80	0.0	0.000	0.00	0.00	0.7	0.429	+4.79	+5.58
0.6	1.027	−4.45	+0.86	0.1	0.001	+0.13	−0.18	0.8	0.242	+2.69	+3.15
0.5	1.155	−0.53	−1.43	0.2	0.139	+1.71	+1.52	0.9	0.073	+0.80	+0.95
0.4	1.083	+2.35	−4.52	0.3	0.333	+3.77	+4.06	1.0	0.000	0.00	0.00

[23] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 1127, Fig. 18.

The structure is analyzed for a unit vertical load at the tenth points,[a] and it is assumed that the base of the pier and the abutments are considered absolutely fixed. In Tables 4 and 5 the sign conventions adopted by the author were used since they are very simple to follow; but in Table 6 the Whitney sign convention was used. On comparing values in Table 6 with what Mr. Whitney[24] gives, it is seen that the two methods agree rather well, and for the two-span arch on an elastic pier the Hrennikoff method is a quick solution for the moments and thrusts.

As reinforced concrete arches, with either fixed or multiple spans on elastic piers will be used more in the future, it seems that some short exact analysis such as that offered in this paper should be applied to multiple arches.

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[24] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), Table 11, Example I, pp. 1118–1119.

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DISCUSSIONS

A DIRECT METHOD OF MOMENT DISTRIBUTION

Discussion

BY MESSRS. G. S. SALTER, LEON BLOG, AUSTIN H. REEVES, E. J. BEDNARSKI, JOHN T. HOWELL, AND I. OESTERBLOM

G. S. SALTER,[24] ASSOC. M. AM. SOC. C. E. (by letter).[24a]—Several papers have been presented recently, on various modifications of the direct method of moment distribution. These variations are interesting if for no other reason than that they show that the profession is becoming increasingly conscious of the adaptability of the moment-distribution method of analysis as presented by Professor Cross, and as designers become better acquainted with the original they devise modifications which they find adapted to their particular use. Too often, most of these simplifications and "short cuts" are such only to the person who devises them and other analysts are slow to adopt even the best of the methods because they prefer to use those with which they are more familiar even though they may be more laborious.

For frames subjected to several conditions of loading or other complexities, time could probably be saved by the use of the author's or some similar method, but for the usual frame and loading conditions, successive balancing of the joints is as simple as could be desired and involves less chance of error. It may be, as Mr. Lin states, that the special value of his modification is its application to continuous beams of varying section; but if so, it would seem that the paper would have been of much greater value if this application had been demonstrated. This probably would necessitate the inclusion of several tables or diagrams before it would have been usable.

Mr. Lin's concept is similar to that offered by Professor Cross[18] in 1929. It is more complete in that Mr. Lin uses modified carry-over factors as well

NOTE.—The paper by T. Y. Lin, Jun. Am. Soc. C. E., was published in December, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1935, by Messrs L. E. Grinter, W. H. Huang, Felix H. Spitzer, Harold E. Wessman, Egor P. Popoff, and L. T. Evans.

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[24a] Received by the Secretary January 21, 1935.

[18] "Exact Method of Analysis," in his paper, entitled "Continuity as a Factor in Reinforced Concrete Design," presented before the Am. Concrete Inst.

as the modified $\dfrac{I}{L}$ (K)-values; but Professor Cross secures the same results by using the regular carry-over values of $\frac{1}{2}$ and by distributing further at successive joints.

As one application of his method Mr. Lin presents the analysis of a symmetrical closed frame (Example 4). Such a frame (which is of common occurrence in the design of water and sewage treatment plants) can scarcely be included in the category of structures which are of sufficient complexity to warrant the use of the author's method because more time would be required in determining the frame constants than would be necessary to make

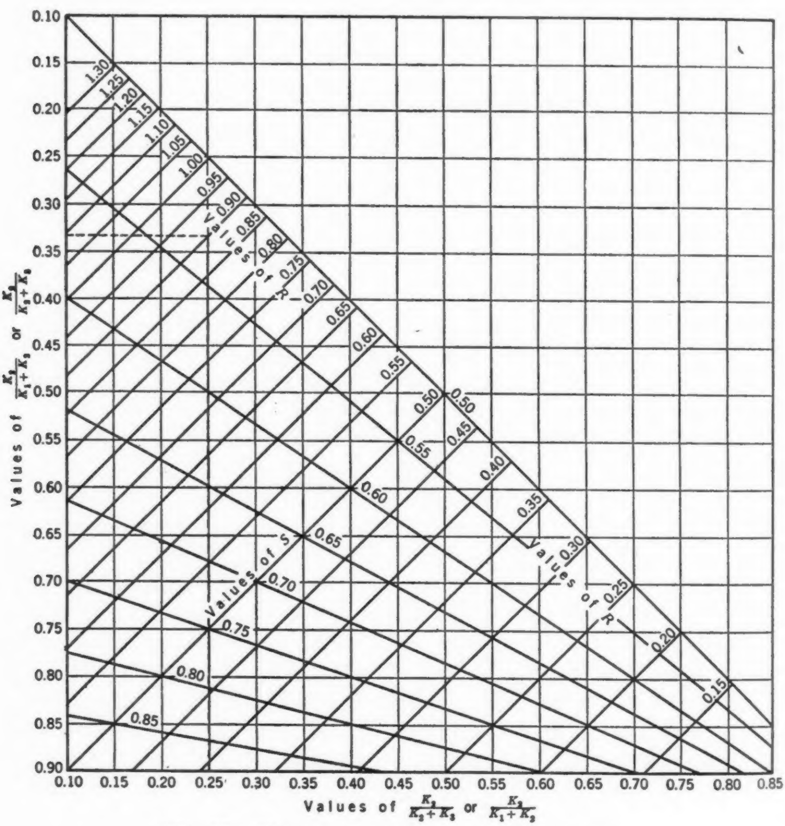

FIG. 19.—MOMENT DISTRIBUTION DESIGN CHART.

the regular moment distribution. However, he could have simplified the analysis materially by utilizing a chart similar to Fig. 19 for the determination of necessary constants.

For example, in the regular moment distribution for a symmetrical frame it may readily be seen that, if the first balancing moments at the two ends of the side members are equal (and opposite in sign) the distribution is

complete. Therefore, it is only the difference between these first balancing moments which calls for further distribution. The following steps will serve as an outline of subsequent procedure (with reference to Fig. 1):

(1) From the K-values of the frame determine the distribution factors for the side members, and from these factors the balancing moments at A and D. Designate these moments as $F = \frac{K_2}{K_1 + K_2} (M_{ab} - M_{ad})$ and $N = \frac{K_2}{K_2 + K_3} (M_{dc} - M_{da})$, respectively. This follows the regular moment distribution procedure. The variation is, as follows:

(2) With the distribution factors as found in Step (1) for the side member enter Fig. 19 using either factor as the abscissa and the other as the ordinate so that their intersection is within the diagram area. At their intersection find the value, S (a stiffness factor), and R, the ratio of stiffness factors.

(3) Multiply by Ratio R the algebraic sum of the values found for F and N. Designate this as B.

(4) Multiply by Factor S the value of B found under Step (3) and this is the corrective moment which applies to that end of the side member the distribution factor of which is the smaller. B minus this value is the corrective moment at the other end of the side member.

The determination of signs for these corrective moments is, as follows: When the balancing moments at the ends of the side members are of the same sign, both of the corrective moments are also of this same sign. When the balancing moments are of opposite sign, both of the corrective moments are of the same sign as the larger of the balancing moments. The writer prefers the sign convention that a moment at a joint is positive when it tends to rotate the joint in a clockwise direction.

The application of the method to Example 4 of the paper is as follows:

By Step (1), $F = \frac{1}{1 + 3} (1\,000 - 0) = + 250$, and $N = \frac{1}{1 \times 2} (0 - 0) = 0$. The corresponding distribution factors are 0.250 and 0.333, respectively.

Under Step (2) enter Fig. 19 with the distribution factors of 0.250 and 0.333, and at their intersection the values of R and S are found to be 0.90 and 0.53, respectively.

By Step (3), $B = 0.90\,(250 - 0) = 225$; and, by Step (4), the correction at Joint $A = 0.53 \times 225 = 119$, and the correction at Joint $D = 225 - 119 = 106$.

Consequently, the balanced moments are, $F_{AD} = 0 + 250 + 119 = 369$ ft-lb, and $F_{DA} = 0 + 0 + 106 = 106$ ft-lb.

The foregoing suggestion for simplifying the solution of Example 4 of the paper has quite a limited application since the frame and loading must be symmetrical. It should be of value, however, to designers who have a number

of such frames to analyze. The construction, Fig. 19, is such that after using it a few times the values of R and S may be estimated readily, and the final moments found accurately and in a minimum of time.

LEON BLOG,[25] ASSOC. M. AM. SOC. C. E. (by letter).[25a]—The writer will discuss this paper in two sections both referring to that part of the paper headed "Limitation of the Method."

Section I.—Limitation of the Lin Modification as Applied to Structures for Which Any of the Values of R as Defined by the Author Are Other Than 1 for a Hinged Condition and ∞ for a Completely Fixed-End Condition.—Hardy Cross, M. Am. Soc. C. E., demonstrated the basic method which the author proposes to modify by utilizing a structure that has its reaction ends of members either truly hinged, truly fixed, or truly cantilevered.[26] There were no approximations as to the degree of restraint in any of the members. In his examples, Mr. Lin makes assumptions as to the degree of restraint of the second-story ends of the columns in Fig. 8 and for one end of the upper horizontal member of the closed frame of Figs. 12 and 13.

To check the effect of such restraint assumptions upon the accuracy of the solution for moments in the structure as well as the accuracy of the assumptions which must be made in order to begin the solution with values of R reasonably close to the true ones, the writer performed a parallel

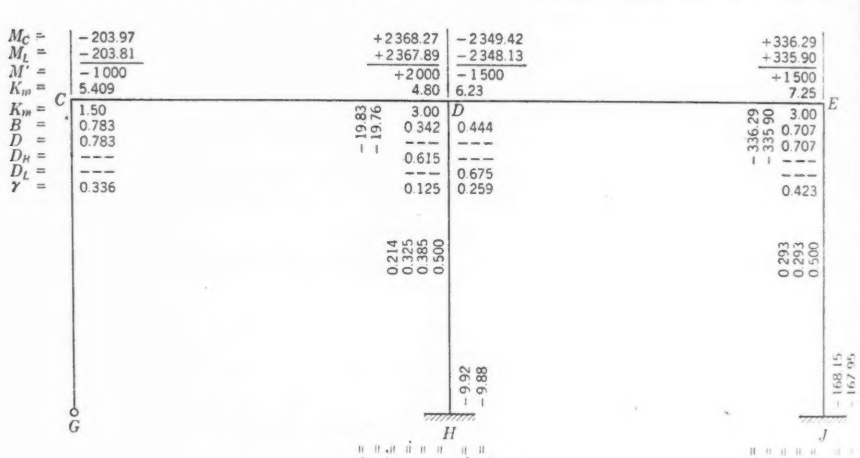

FIG. 20.—COMPARISON OF STATIONS, PART OF FIG. 8.

distribution of a moment of 100 at D in Fig. 9 by giving R_{ad} and R_{be} simultaneous values of 3.5 and 6.5, and repeating the distribution with values of 2.5 and 5.5. The significant resultant moments are given in Fig. 20, and the relative values stated in percentages of the author's moments assumed

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[25a] Received by the Secretary February 6, 1935.

[26] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 5.

as the true ones are given in Table 1, which also includes the modified carry-over and stiffness factors for End D, of Member DA, for reasons which follow.

From Table 1, observe that the greatest moment variations from the author's values occur for M_{ad}. For the case of restraint factors assumed as

TABLE 1.—RELATIVE VALUES OF CERTAIN MOMENTS OF FIG. 20 (EXPRESSED AS PERCENTAGES) BASED ON THE MOMENTS FOR $R_{ad} = 3.00$ AND $R_{be} = 6.00$

R_{ad}	R_{be}	M_{dc}	M_{dh}	M_{da}	M_{ad}	M_{de}	M_{ed}	γ_{dam}	K_{dam}
3.5	6.5	99.74	99.21	100.98	106.86	99.58	99.79	105.78	101.27
3.0	6.5	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
2.5	5.5	100.91	100.48	98.68	90.43	100.46	100.41	91.48	98.18

0.50 more than those of the author, the surplus for M_{ad} is roughly 6% and for the case of 0.50 less than the author's, the deficit is about 9 per cent.

For the restraints assumed by the writer, the modified stiffness factor, K_{dam}, does not vary much from Mr. Lin's value so that K_{dam} cannot be the explanation for the large variations in M_{ad}; but the modified carry-over factor, γ_{dam}, varies almost as much as the moment, M_{ad}, and is almost directly responsible for the variations from the author's value of M_{ad}.

Inspection of Equation (4) reveals that when the restraint assumed at the remote end of a member is too low the modified carry-over factor of the near end of the member will be too low. An assumed high restraint will yield a high carry-over factor. Correspondingly, the moment at the remote end will be too low or too high.

With the aid of Equation (1), the writer examined the accuracy of the assumptions which must be made in order to begin the solution for moments with values of R reasonably close to the true ones. Equation (1), generalized, is:

$$R = \frac{K + \Sigma K_m}{K} \qquad (24)$$

Equation (24) reveals that the value of R is governed by the combined modified stiffness factors of all the members entering the joint, excluding the member for which R is desired; and it is also governed by the non-modified, or Cross stiffness factor. These modified stiffness factors, in turn, are dependent upon the restraint factors assumed or derived for the remote ends of those members. Finally, for a multi-story building, the degree of restraint of the ends of members forming the ceiling of the top story must be known. The limitation of the author's modification now appears. Unless at the outset of the analysis of such a structure, he can determine with certainty the degree of restraint that will prevail in the top-story members, the writer is unable to understand how Mr. Lin can claim that he has presented a rigid method having the same limitations as that presented by Professor Cross.

It is true that similar definiteness as to the degree of restraint or fixity for the ceiling members of the top story is requisite to the analysis of a multi-storied structure by the pure Cross method; but Professor Cross did not find it necessary to introduce the concept of restraint factors intermediate in value between a hinged condition and complete fixity which the author evaluates, respectively, as $R = 1$ and $R = \alpha$. Having introduced the concept of partial restraint in connection with the Cross method, modified, Mr. Lin should give some indication as to the accuracy of estimating such restraint.

TABLE 2.—ALLOWABLE ERROR IN OBTAINING COMBINED MODIFIED STIFFNESS FACTORS OF EQUATION (1)

R_{ad}	K_{ad}	K_m of adjoining members	Allowable error	R_{be}	K_{be}	K_m of adjoining members	Allowable error
2.5	6	9	....	5.5	2	9	....
			3				1
3.0	6	12		6.0	2	10	
			3				1
3.5	6	15	....	6.5	2	11	....

Table 2 shows how much error could be allowed in obtaining the combined modified stiffness factors of Equation (1) for the values of R which the writer assumed in his solution of Fig. 9, whereby he obtained the moment values of Table 1. Whether it is possible to stay within the limits of this error for a multi-storied structure, the writer is not prepared to state.

He has checked the closed structure of the author's Fig. 12 and Fig. 13 with the aid of a method presented[27] by L. T. Evans, Jun. Am. Soc. C. E. This is a method of determining the angular rotations produced in a structure when a unit moment is applied at one end of a member. However, it involves the idea of restraint at the end of the member opposite the one at which the unit moment is applied and is thus similar in that respect to the author's concept which is based on the amount of moment or stiffness which will produce a unit moment at the rotating end while the other end remains fixed. Mr. Evans gives a table by means of which such restraint may be estimated with reasonable precision. Were Mr. Lin to develop his modification so as to permit an estimate of the restraints which are considered the limitation of the method, the writer would use it for multi-storied structures for reasons which are enumerated in Section II of this discussion.

Section II.—Application of Lin Paper to Such Structures for Which the Reaction Ends of Members Are Definitely Either Hinged or Fixed-Ended and for Which, Therefore, the Values of R, According to the Author, Are Respectively, 1 and ∞.—Fig. 21 shows a part of Fig. 8, the upper story columns and Span *EF* having been deleted. The symbols, including those introduced by Mr. Lin, are defined as: M_C = moment by the Cross method, without simplifications; M_L = moment by the Lin modification, according to Example 3; M' = fixed-end moment; K_m = stiffness factors; B = balancing factor applied to an unbalanced moment derived from the fixed-end moments,

[27] "Handbook of Rigid Frame Analysis," by L. T. Evans.

M'; D = distribution factor; D_R = value of D to be applied to an unbalanced moment which occurs to the right of Column DH; D_L = value of D to be applied to an unbalanced moment which occurs to the left of Column DH; and γ = modified carry-over factor. The members of this

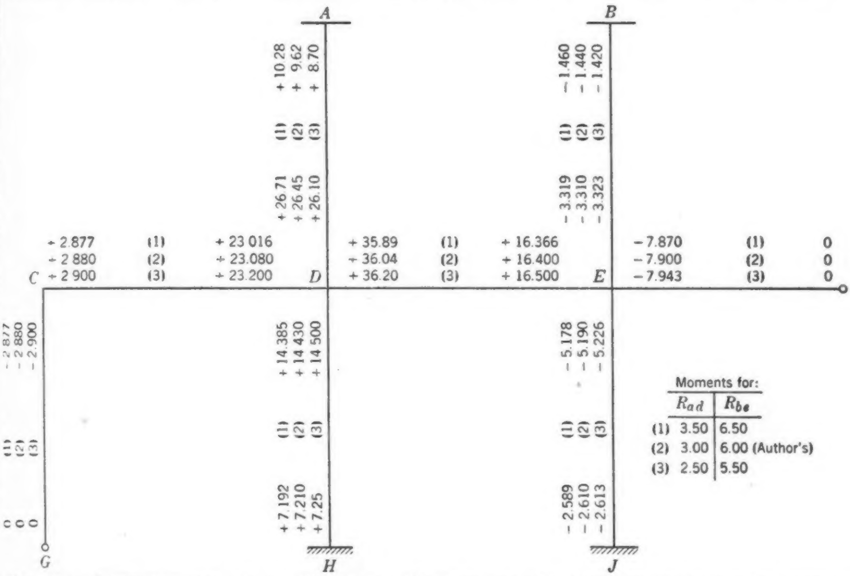

FIG. 21.—DISTRIBUTION OF AN EXTERNAL MOMENT OF 100 APPLIED AT D USING STEP 2 OF THE LIN PAPER.

abbreviated structure have been given the same unmodified stiffness and carry-over factors, and the assumed fixed-end moments distributed are the same as those in Fig. 11. The restraints assumed for the reaction ends of the lower story columns are the same as those of Fig. 11. Since there is no upper story, the question of estimating the restraint of any member does not enter the solution of the problem. A comparison between the Cross method, without simplifications, and the author's modification can then be made. Fig. 21 shows how closely the two solutions check each other, utilizing an ordinary slide-rule.

TABLE 3.—COMPARATIVE WORK INVOLVED IN SOLUTION OF FIG. 21

Method	Preliminary distribution ratios and carry-over factor calculations	Items written in solution	Total steps	Balancing cycles
Cross........	7	120	127	6
Lin..........	36	21	57	1

Table 3 shows the relative labor involved in these solutions. The number of written items could have been reduced to 108 by using $\gamma = 0.75$ for M_{cg}, thus eliminating twelve items at End G in the Cross solution. The total number of steps in the Lin solution would then be about one-half that of the simplified Cross method.

The preliminary work for the Lin solution is about five times as great as that of the straightforward Cross solution and involves somewhat more care. It should also be noted that the distribution ratio as defined by Professor Cross and used throughout the Cross solution is used only once in the Lin solution. The writer chooses to term this the "balancing factor" because by its use a "balance" is created between the moments at the ends of all the members that enter the joint which was previously subjected to the action of unbalanced moments. The next distribution of moment at Point *D* requires the use of a different distribution ratio than the balancing factor. The author has fully developed the calculation of this ratio, but did not dwell on the fact that a column will have two secondary distribution ratios. A moment which has been carried over from Joint *C* to Joint *D*, Fig. 21, must be distributed to Member *DE* and Column *DH*. "Carried-over" moments will occur at both the right and the left of the joint. It is easy to remember that the member which supplies the carried-over moment does not share in the distribution and that its modified stiffness factor is not used when evaluating the distribution factor. The writer has also found it convenient to erase the balancing factor temporarily in order that it shall not be used after the first distribution.

To some readers who have noted that the calculations preliminary to actual distribution will be the most numerous for the Lin modification, the writer indicates the extreme rapidity of convergence of this technique. This quality results in a minimum of distributing and of carry-over items which are quickly added for a result. This part of the solution can be checked quickly. In the case of the pure Cross method, should the initial fixed-end moments be large or, should the convergence be slow, periodic summations of numerous moments must be made to determine when to cease distributing. Through an inadvertent interchange of distribution ratios between two members, joint moments may be erroneously balanced. This will not be discovered except through common-sense check when possible; but the writer prefers a technique which permits of speedy checking. He believes that the Lin procedure, when based on definitely determined restraint factors, affords such speed.

In conclusion, this modification of the Cross method is commended to the structural engineer for one-story structures with truly fixed or pin-ended reaction ends of members. If the author will develop a rapid method of estimating restraints for intermediate degrees of fixity, he will have made a valuable labor-saving contribution toward the solution of framed structures by the Hardy Cross method.

AUSTIN H. REEVES,[28] Assoc. M. Am. Soc. C. E. (by letter).[28a]—The "Conclusion" of this paper contains this sentence: "However, it is not recommended as an invariable substitute for the original Cross method."[2] It is neither

[28] Newark, N. J.

[28a] Received by the Secretary February 25, 1935.

[2] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

advantageous nor advisable to substitute the author's method for the Cross method; the probable outcome of such substitution would be confusion.

When thoroughly understood, the Cross method is applied easily to any type of frame with the greatest simplicity, speed, and accuracy. The writer was much better able to utilize the Cross method because of a previous working knowledge of a few other methods, the most helpful of which was the conjugate point method.[29] This helpfulness was especially evident in the design of frames containing girders or columns with variable moments of inertia. A designer of rigid frames of all types is almost perfectly equipped if he has a thorough understanding of the application of the original Cross method, supplemented by an equally good working knowledge of the conjugate point method.

Mr. Lin could have explained his method much more easily by a statement that it is one of the special cases of the conjugate point method adjusted in a new and more complicated form to appear as a modification of the Cross method.

For example, to find the author's carry-over factors for any span, AB, by the conjugate point method, note where the U-line (Fig. 22) cuts the

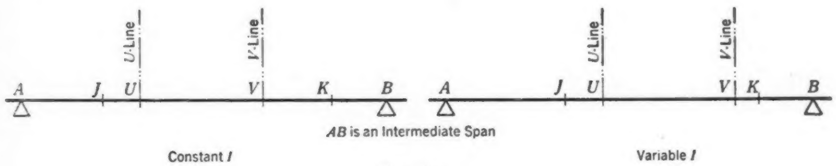

FIG. 22.

span at Point U and where the V-line cuts the span at Point V. Now, $-\dfrac{AU}{UB} = \gamma_{ba}$ and $-\dfrac{VB}{VA} = \gamma_{ab}$. To find the author's "modified" carry-over factors for any span, AB, note where the J and K fixed points are located. Then, $-\dfrac{AJ}{JB} = \gamma_{bam}$ and $-\dfrac{KB}{KA} = \gamma_{abm}$. After obtaining γ_{abm} and γ_{bam} for all members, it is easy to find the author's K_{abm} and K_{bam} in each case. There would then be no necessity for calculating or tabulating R_{ab} and R_{ba} as the author has done.

This leads up to what the writer considers the most unfortunate feature of the paper; namely, the signs. The carry-over factors should not be positive, but should have negative signs. To prove this statement, consider a horizontal continuous beam of three spans with simply supported ends. Twist the right-hand end with a moment, M, in a vertical plane through the longitudinal axis of the beam, there being no other loads on the continuous beam. The moment closing line will pass leftward from Point M, distance up or down on the R_4 line, and through J_3 until it intersects the reaction line, R_3; from this point on R_3 it will pass through J_2 in the middle span

[29] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 1.

until it hits the reaction line, R_2; from which last named point it will pass through J_1, which is on the R_1-line. This demonstrates clearly that when a moment closing line passes through a point of inflection, the sign of the moment changes either from plus to minus or from minus to plus. Therefore, any carry-over factor should have a minus sign.

Furthermore, to finish a problem as in Fig. 11, Corner *C*, and have + 127.4 on one side of the joint and − 127.4 on the other side, is at variance with both American and foreign texts, with customary practice (except in the slope deflection method),[30] and with common sense. It has been a source of deep regret to the writer that so many brilliant professors and engineers have seen fit to reverse Professor Cross on the matter of signs.

The writer agrees with Professor Cross as regards signs except in the following slight deviation in complicated frames; namely, the writer looks at all girders, truss chords, or arches from the bottom, at all outer columns from inside the structure, and at interior columns or web members of trusses from either direction. The moments are written on any member of a framework in such a manner that there can be no error as to which way the designer faced the member.

After solving numerous complicated problems of various types by the Cross method, the writer believes the Cross sign conventions eventually will be in general use, except for minor variations. In a paper entitled, "Sign Conventions for Bending Moments in Rigid Frames,"[31] Mr. Robins Fleming, of the American Bridge Company, states: "Notwithstanding the vigorous exceptions that have been taken, the writer has a preference for the convention used by Professor Cross * * *."

Some of the disadvantages of Mr. Lin's modification are:

1.—Considerable preliminary work is required before a beginning can be made on the actual determination of moments;

2.—Side-sway, when present due to (*a*) unsymmetrical loads; (*b*) an unsymmetrical frame; or (*c*) the more apparent overturning moments, would further increase the necessary preliminary work;

3.—More concentration of thought is required during the determination of moments, due to the necessity of handling two rigidities and two carry-over factors in each span instead of one;

4.—It is subject to erroneous interpretation by any one but an expert designer; and

5.—The work necessary to obtain the final moments increases rapidly with the number of members that are loaded.

It seems proper to conclude by using the fifth disadvantage to disprove one of the author's concluding statements: "The greatest advantage of this method is its simplicity and directness." Fig. 10 appears simple, but only one member was loaded and to the labor on this simple drawing must be added the work of preparing the data on Fig. 8. Fig. 11, with three members of this same frame loaded, certainly is much more complicated than

[30] *Bulletin No. 108,* Univ. of Illinois, Eng. Experiment Station, Urbana, Ill.

[31] *Engineering News-Record,* February 14, 1935, p. 258.

Fig. 10 and, furthermore, the data on Fig. 8 are necessary for the preliminary analysis. Imagine how Fig. 11 would look if each of the eight members had been loaded. It would then be highly involved; but in addition it would have charged against it the time spent on Fig. 8 which contains a mass of data. More than one-half the data on Fig. 8 is unnecessary in the original Cross method, and it is this unnecessary majority which requires a large amount of work to produce. Besides being a highly involved method, lacking in simplicity, it is not a direct method. Imagine, in a speed contest, two designers given a complicated rigid-frame problem which neither had seen before with all necessary data to solve it by the original Cross method, but with no limitations on the method of solution. Assume that, Designer No. 1 elects to stake all on the author's method whereas Designer No. 2 chooses to stake all on the original Cross method. At the expiration of 30 min of calculations, a halt is called and each contestant is then informed that there will now be available only 5 min in which to estimate on the basis of the work already done the final size and sign of all end moments. Designer No. 1 probably would be in no position to make this estimate as he would scarcely have finished with all the preliminary data such as those shown on Fig. 8. On the other hand, Designer No. 2 probably would be about through with at least the second balancing and could make a fairly accurate forecast of the answer. This becomes important in design work where it may require two or more trials before the final moments of inertias of the members can be determined. Thus, which method would seem to be more simple and more direct?

E. J. BEDNARSKI,[32] ASSOC. M. AM. SOC. C. E. (by letter).[32a]—The classical universal method requires a solution of n simultaneous equations with n unknowns in each equation. When the value of n is fairly large—as, for example, 9—the work required is theoretically 60 480 times more than is necessary for the solution of one equation with one unknown. This gives an approximate idea of the work necessary in computing the stresses by the classical method in a multiple-story structure with several bays, such as the Empire State Building, in New York City.

Thus, the development of methods of analysis which save work is of great importance. The author solves the problem by developing original formulas which are simple and constitute the inherent properties of a given structure and which permit the distribution of any moment around any joint of the structure and the reaction to this moment by the other end of this member. Development of these relations is the most important and valuable result of the author's endeavor. Actually, the determination of the values of the factors, K_m and γ_m, constitutes the essence of the computation. (In certain structures the process of determination of K_m and γ_m will require one, or not more than two, repetitions of the computation for all practical purposes.) They are valid for any loading, in any structure that does not sway sidewise Unfortunately, in the analysis of ordinary structures, lateral forces and un-

[32] Structural Engr., San Francisco, Calif.

[32a] Received by the Secretary March 1, 1935.

symmetrical loads that act on a structure which is not capable of resisting side-sway, are the most common cases encountered. The method is an excellent one, however, as long as its application is limited to symmetrical structures under symmetrical loading, continuous portals secured against side-sway, and continous beams. The writer checked a continuous portal given as an example by the late Professor A. Ostenfeld[33] and found the results identical with those given in the paper.

In a simple structure, such as Example 1, an application of other methods may prove just as efficient. The force polygon method introduced by Mr. J. D. Gedo, for instance, has certain advantages in producing a direct exact answer to the problem by means of the solution of two simple simultaneous equations.[34] This corroborates the statement made by Professor Ostenfeld in his life work on statically undeterminate structures,[35] that each particular problem may be solved best by a certain particular method, although it may require a special study in itself.

The development of the "short cuts" in the analysis of statically indeterminate structures has recently gained a tremendous impetus. The writer wishes to express his hope that Mr. Lin may develop a solution of the problem that will include lateral and unsymmetrical forces acting on structures not secured against side-sway.

John T. Howell,[36] Jun. Am. Soc. C. E. (by letter).[36a]—The method proposed by the author has proved to be of real practical value in the analysis of continuous frames. Engineers familiar with Professor Cross' method of moment distribution will be pleased with the manner in which the original concepts and definitions have been used to set up a direct method for determining moments. For those not yet familiar with this direct method a short general explanation will be given as the best way of comparing it with other modern methods.

The method offered is based entirely on Professor Cross' definitions of stiffness, carry-over factors, and fixed-end moments; and in a straightforward ingenious manner Mr. Lin develops simple relations for "modified" stiffness and carry-over factors.

The usual fixed-end moments are then distributed and "carried over" quite as one would visualize the restraining action of the actual frame. This "travel" of a moment to the ends of the frame is novel and interesting. Such a procedure is of importance to the designer as it affords valuable training in the art of estimating, accurately, the proportion and distribution of moment in any certain part of a structure, as controlled by the sizes and shapes of component members. This is true, since final moments are not determined by equations or formulas, or by repeated distributions, but are "written" on the frame in a regular routine manner by the use of simple arithmetic.

[33] *Der Eisenbau*, November 8, 1921, p. 285.

[34] "Theory of Superstatic Structures," by J. D. Gedo, New York, 1935, Equations 117 and 118, p. 46.

[35] "Die Deformations Methode," by A. Ostenfeld, Berlin, 1926, Verlag Julius Springer.

[36] Junior Designing Engr., U. S. Bureau of Reclamation, Denver, Colo.

[36a] Received by the Secretary March 5, 1935.

The method may be said to be of a type between the original method of moment distribution and various methods in which final moments are obtained directly by formulas. The analyst who prefers the simplicity of moment distribution and its easy application, but who wishes to avoid the numerous cycles of distribution necessary in many cases, will find great use for the direct method. After some practice one may estimate, with accuracy, the modified beam factors from the original stiffness and carry-over factors in particular cases. It is thus possible to "write in" the final moments in members of such indeterminate structures as multi-storied building frames with a degree of accuracy that is usually sufficient.

The most interesting feature of the method provides the greatest advantage or disadvantage, as the case may be, in comparing it with the method presented by Professor Cross. Since modified beam factors are determined before the fixed-end moments are distributed, and this preliminary calculation usually requires the major portion of the total time of analysis, it is important to consider the purpose and extent of any given analysis. In those cases where there is only one load, or a group of different loads taken together, it is quicker to use ordinary moment distribution. However, in cases where it is preferable to study the load effects separately, and in particular when influence lines are to be plotted, the direct method is much quicker and more satisfactory. This comparison shows that each method has its own field of use. The writer has found from experience that Professor Cross' method is quicker, and, of course, easier to apply in cases where one or two sets of joint moments are desired. The sign convention commonly attributed to L. E. Grinter, Assoc. M. Am. Soc. C. E.,[37] and also his "short-cut" in which large unbalanced moments are distributed and carried over before starting the regular distribution cycle, save some time.

In other types of detailed analysis previously mentioned Mr. Lin's modification is thought to be preferable. Considerable time will also be saved if the two fundamental equations for the modified beam factors, K_{abm} and C_{abm} (Equations (3) and (4)) are set up in the form of nomograms.

The writer has analyzed 8, 9, and 10-bbl siphon structures by both methods, and has found that in the calculation of moments due to several load conditions where such effects are considered separately, and for the necessary analyses for shear correction coefficients, Mr. Lin's method is not longer than that of Professor Cross. A little practice in the determination of the "modified" beam factors for such frames will prove the procedure far easier than anticipated.

On the surface Mr. Lin's method and the alternate Cross method in which the "end-rotation constant"[38] is used, would seem somewhat similar. This is true in a superficial sense only, however, as the author's method is perfectly general whereas in the reference given "the end-rotation constant" method can only be used in those particular cases where the ratio of the changes in moment at the ends of the members are known.

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[37] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 610.

[38] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, Members, Am. Soc. C. E.

During two years of practical use of the author's method the writer has noted a few points which might be mentioned. Under the heading "Definitions and Notation" the author has defined K as the stiffness of the end of a member. "Relative stiffness" values are most commonly used, the "absolute stiffness," that is, $K \propto \frac{E\,I}{L}$ being used where different materials make up the structure, or in the determination of fixed-end moments of haunched members subject to the lateral displacement of one end relative to the other, with no joint rotation.

Stiffness values in published tables or curves are usually in terms of $\frac{I}{L}$ or $\frac{4\,I}{L}$. In general, any numerical value may be given to the K-values as long as they are consistent and proportionate to the respective $\frac{I}{L}$ -values. The modified stiffnesses (Equation (3)) change by this same proportion, and the constants are cancelled in Equations (1) and (4).

To clarify any possible slight confusion, it should be noted that in Mr. Lin's terminology the carry-over factor placed at an end of a member is that one which carries over distributed moments to the opposite end. In many methods the opposite is true.

The actual degree of restraint provided at the joints of a structure has been the object of much research in recent years. Riveted connections of various types have been shown to have different restraint ratios.[39] To obtain the most accurate analyses, correlated and consistent experimental data will be used to indicate the "degree of fixity" provided at the ends of critical members. For such analyses Mr. Lin's method is ideal. The R-term (Equation (1)) denotes the end restraint of a member, and may be assigned any value indicated by such research. The modified beam factors and, lastly, the joint moments are then found as before.

The fundamental formulas in the direct method may be further modified to make them correspond to formulas in other methods. The final moments at the ends of a member can be given by those altered formulas, but it is the opinion of those familiar with the author's method that one of its chief advantages lies in the fact that it retains some of the features of moment distribution.

The writer found the direct method easy to learn, and its application has afforded real pleasure. Mr. Lin has made a worthwhile contribution to structural engineering literature.

I. Oesterblom,[40] M. Am. Soc. C. E. (by letter).[40a]—This paper seems to have its principal value in demonstrating that a framework has elastic properties entirely apart from any system of loads which may be applied

[39] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, M. Am. Soc. C. E *Proceedings*, Am. Soc. C. E., January, 1935, p. 3.

[40] Civ. Engr., Chicago, Ill.

[40a] Received by the Secretary March 19, 1935.

(in the same manner as a beam), and that these properties are definitely ascertainable by simple and elementary analysis.

This is not to say that Mr. Lin is the first to develop this idea. Presumably, it has been studied by many investigators, but no one has stressed the significance sufficiently to cause much quotation or comment. An exception might be Dr. Hermann Zimmermann, who long ago saw the significance of segregation and presented papers on the subject as early as 1907, before the Akademie der Wissenshaften,[41] in Berlin, Germany, and other scientific bodies. Unfortunately, Dr. Zimmermann did not realize that his many contributions assembled into one unit would serve as a powerful tool of analysis and design, until quite recently, and when his book on "Knickfestigkeit der Stabverbindungen" was published in 1925, it caused little or no comment.

To Mr. Lin, therefore, belongs the honor of bringing before thinking engineers an important fact, and, if anything, it merely adds to its value that the presentation has been made in terms of Professor Cross' brilliant moment distribution concept.

It is to be regretted, however, that Mr. Lin did not confine himself to "elastic properties of a framework." That subject is so broad in itself as to provide material for many an academic thesis, and it needs emphasis because it has been so sadly neglected as a separate issue.

On the other hand, the writer is not convinced that Mr. Lin's procedure for finding the moments due to a system of loads will lead to results as quickly as Professor Cross' original procedure. For example, Equation (1) is most excellent for the analysis of important structures, where it is necessary to show graphically and clearly the elastic relations of the whole, but it should not be forgotten that in the usual framework four such equations must be solved for every nodal point. This means considerable work and is only a part of the total analytical task. The time required as compared to that allowable, for the usual run of designs, would automatically bar the method.

Professor Cross aimed at a series of progressive approximations leading ultimately to any degree of mathematical accuracy and all of a nature to be performable, even if not performed, by slide-rule. In ruling out the progressive approximations and bringing back a series of equations, it would seem that, on that point, Mr. Lin is taking a step backward. This criticism, however, is directed only toward over-enthusiasm in regard to the most vulgar applications and in no way lessens the writer's admiration for the fundamental idea of the paper.

In view of its importance the paper is regrettably brief and for that reason is not fully convincing at all points. Unfortunately, most engineers would like to examine the mathematical demonstrations in full, and thus avoid spending extra time to provide "the missing pieces in the game." A few more pertinent details would have been helpful.

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[41] *Sitzungsberichte*, 1907, 1909, 1921, 1923, 1924, and 1925.

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DISCUSSIONS

ELASTIC PROPERTIES OF RIVETED CONNECTIONS

Discussion

BY MESSRS. HAROLD C. ROWAN, WALTER SCHOLTZ, J. F. BAKER, L. E. GRINTER, AND C. R. YOUNG AND K. B. JACKSON

HAROLD C. ROWAN,[7] ESQ. (by letter).[7a]—The necessity for information as to the rigidity of steel-frame building connections has been widely recognized by structural engineers. In reporting the results of these tests on standard connections the author has provided much useful information.

In the "Introduction" it is stated that no extensive tests have been published on this type of investigation. The writer would like to call attention to the work of the Department of Scientific and Industrial Research of Great Britain.[8] A report published in May, 1934, contains, among other material, the results of a comparatively extensive series of tests conducted at the University of Birmingham, under Professor Cyril Batho. The writer participated in the work.

There appears to be no serious discrepancy between the results obtained by the author for the moment-deformation curves and those obtained at Birmingham. The manner in which flexible connections failed is meaningless because it is impossible in practice for even a small fraction of the deformation involved in failure to occur. The only possibility of failure occurring in such connections is by fatigue. Although it may be true that the strength of the specimens in Series *A* and *B* in shear is not impaired by considerable flexure under moment, their manner of failure alone does not justify that conclusion. Furthermore, the effect of shear on connections of Series *B* is to increase the rigidity. This point may be quite important when considering the rigidity of a dual connection, such as Specimen 11.[9]

NOTE.—The paper by J. Charles Rathbun, M. Am. Soc. C. E., was published in January, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1935, by Ralph E. Goodwin, Assoc. M. Am. Soc. C. E.

[7] Commonwealth Fund Fellow, Univ. of Illinois, Urbana, Ill.

[7a] Received by the Secretary February 16, 1935.

[8] Second Rept. of the Steel Structures Research Comm. of the Dept. of Scientific and Industrial Research, pub. by His Majesty's Stationery Office, London, England, 1934.

[9] *Loc. cit.*, p. 93.

The author remarks that the most serious stresses occur in the lower seat angles of Series *B*. It would be noted that the connections were tested upside down as compared with their position in a building, and it is really the top angle which is highly stressed and results in most of the deflection. The same applies to the rivets in the vertical legs of the angles. Specimens 8, 9, 10, 11, and 12 should possibly have been tested the other way up.

The correctness of the author's inference that a seat-angle connection may be stiffened materially by increasing the thickness of the angles was clearly demonstrated, within reasonable limits, in the Birmingham tests.

The type of specimen in which a single plate is used instead of a column section is not representative of the actual conditions obtained in a building. On account of the fact that the column rivets are common to both sides of the specimen, the connections behave as if a rigid plane existed at the mid-section of the plate. When the connection is to a column flange, this is not true, and flexure of the flange will cause lower rigidity. This effect will be largest in the wind-bracing connections, especially where four rows of rivets are used.

The method given by the author for the solution of semi-rigid frames using constants to represent the degree of rigidity of the connections is interesting. It may be mentioned that the British report[8] gives three methods of solution using the same assumptions. A method of interpreting results which does not involve the straight-line assumption, and in which the moment-deformation curve is used to yield immediate results without resorting to trial and error, is also given in the report.[10] By its use the end moments resulting from a uniform load on a beam having the connections under investigation are thus found by drawing a single straight line on the experimental curve and reading the fixing moments directly from the graph. For purposes of interpretation it is best to assume that the beam is attached to rigid columns, but the method is also extended to include the effect of flexure in surrounding members, assuming that their rigidity may be estimated approximately. The effect of a change of load, change of stiffness of a beam due to altering its length or moment of inertia, change of stiffness of surrounding members, or the substitution of some other connection, can be seen immediately. The method has since been extended to cover the cases of unsymmetrical loading and unequal end conditions, or both.

The rapid design of steel-frame buildings, which is so often necessary, renders it highly improbable that any method involving complete analysis can ever be used in design, and certainly not any complete analysis using trial-and-error methods. Although it is possible in any single case to choose a constant for the moment-deflection relation which will give the same resulting moments as a curve, it may be shown that any such method involves a change in that constant for different lengths of beam, different loads, and different surrounding members. In any method, of course, the accuracy cannot be greater than the reliability of the connections to act according to the assumed or experimental graph, and tests have shown that the curves extend over a fairly wide range.

[10] Second Rept. of the Steel Structures Research Comm. of the Dept. of Scientific and Industrial Research, pub. by His Majesty's Stationery Office, London, England, 1934, p. 92.

There are other considerations which make any analysis approximate. The most important is probably the fact that the members and connections of a steel-frame building are generally encased in concrete, and this will probably affect the distribution of the moments and the stresses. Another uncertainty is the capacity of a building to sway sidewise under unsymmetrical loads. Walls, partitions, and floors tend to stiffen the frame, and it is possible that an analysis which neglects side-sway more nearly represents the actual conditions than one which takes account of side-sway.

It may be mentioned that a large number of tests have been made at Birmingham since the writing of the second report mentioned previously.

WALTER SCHOLTZ,[11] JUN. AM. SOC. C. E. (by letter).[11a]—In the solution by moment distribution presented in the Part II of this paper, modified slope deflection equations were used to obtain expressions for fixed-end moments, distributed moments, and carry-over factors. The labor of solving these somewhat complicated expressions could be greatly simplified. For example, referring to Fig. 31, consider the elastic connection as an additional, or phantom, member in the frame. For the connection in this example the stiffnesses given for each side are: Left side, $\dfrac{1}{Z_L} = 2.9 \times 10^8$; and, right side, $\dfrac{1}{Z_R} = 4.2 \times 10^8$. For both acting together,

$$\frac{1}{Z} = \frac{1}{Z_L + Z_R} = \frac{1}{\dfrac{1}{2.9 \times 10^8} + \dfrac{1}{4.2 \times 10^8}} = 1.69 \times 10^8,$$

in which $\dfrac{1}{Z}$ is the absolute stiffness, $\dfrac{M}{\theta}$, for the connection as a member. Its carry-over factor obviously is -1. The analysis now can be performed by ordinary moment distribution, the only precaution to be observed being that of utilizing the absolute stiffness for all members so that the correct relationship to $\dfrac{M}{\theta}$ for the connection is established. The ordinary fixed-end moments and the carry-over factor of $\frac{1}{2}$ are used.

(1) Points	A	B_1		B_2		C		D
(2) Length, L	20' 0"		Phantom Member for Connection		10' 0"		10' 0"	
(3) Stiffness, $\frac{M}{\theta} \times 10^{-8}$		2.98	1.69	1.69	2.16	2.16	1.62	
(4) Percentage Distributed		63.8	36.2	43.9	56.1	57.1	42.9	
(5) Carry-over Factor			−1	−1	0.500	0.500		
(6) Fixed-End Moments		−60.00			+10.00	−10.00	+15.00	
(7)	Standard Cross Distribution (Six Cicles)							
(8)		−16.89					+10.78	

FIG. 36

In the solution shown in Fig. 36 the writer has neglected the reduction of the moment at the ends of members due to transfer of shear to the connection at each row of rivets; but the effect of this factor is clearly of small magni-

[11] Graduate Asst., California Inst. of Technology, Pasadena, Calif.

[11a] Received by the Secretary February 19, 1935.

tude. The results obtained are identical with those that would be obtained by using $L_2 = L + 3\ E\ I\ Z$ in the equations given by Professor Rathbun. Instead of the stiffness factor, K, in Fig. 32, the writer suggests the factor of absolute stiffness, KE, or $\frac{M}{\theta}$. For Span B_1A (Fig. 36, Line 3) $\frac{M}{\theta} = 9.93 \times 3 \times 10^7 = 2.98 \times 10^8$; for Span B_2C, $\frac{M}{\theta} = 7.2 \times 3 \times 10^7 = 2.16 \times 10^8$; and, for Span CD, $\frac{M}{\theta} = 5.4 \times 3 \times 10^7 = 1.62 \times 10^8$. (Note that, for the opposite end fixed, $KE = \frac{4\ I}{L}$ and for the opposite end free, $KE = \frac{3\ I}{L}$). The physical picture on which Fig. 36 is based will be visualized easily by any one familiar with the Cross method, and it permits a simple application in design calculations of the valuable data reported in this paper.

It should be noted that the loading used in this example is only a fraction of the allowable load for these beams. Inasmuch as the slopes of the rotation-moment curves for all the connections decrease materially as the load increases, it follows that for design loadings the connection stiffnesses would be materially less than those used in Fig. 36, which were found by taking the slope of the tangent to the rotation-moment curve at the origin.

J. F. BAKER,[12] ASSOC. M. AM. SOC. C. E. (by letter).[12a]—A better understanding of the behavior of steel work connections is essential if the design of steel frames is to be put on a more rational basis and full economy of material obtained. This understanding will follow only from a careful study of such tests as those conducted by Professor Rathbun. It is unfortunate, however, since the behavior of connections is so complex and the field to be covered is so wide, that Professor Rathbun was unaware of the work that has been done in Great Britain since 1930 for the Steel Structures Research Committee of the Department of Scientific and Industrial Research, which was appointed in 1929 on representations from the steel industry backed by an offer of support from the British Steelworks Association. As mentioned in the Committee's First Report, published in 1931, a series of laboratory experiments on beam-to-stanchion connections was undertaken. A detailed report of certain of these tests (which were conducted by Professor Cyril Batho, of Birmingham University, on a series of connections, including all the types dealt with by Professor Rathbun) is given in the Second Report of the Committee, published in 1934.[13] It would be out of place to give a summary of that work here, but it should be studied by all interested in steel design. Characteristics of the connections were given by curves similar to those in the paper under discussion.

Professor Rathbun's analysis of framed structures is clearly presented, although certain of the assumptions made in introducing Equations (15)

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[12] Prof. of Civ. Eng., Univ. of Bristol, Bristol, England; Technical Officer, Steel Structures Research Committee, London, England.

[12a] Received by the Secretary March 7, 1935.

[13] Second Rept., Steel Structures Research Committee, Dept. of Scientific and Industrial Research, pub. by H. M. Stationery Office, Lond., 1934.

to (21), to obtain a more accurate expression for L_2, are of doubtful value. In the First Report of the Steel Structures Research Committee already mentioned, a general equation was given applicable to frames with semi-rigid connections, to which Professor Rathbun's Equations (4), (5), (8), and (9) will be found to lead. The method of attack in stress analysis is largely a matter of taste, but it is interesting to compare this work. The derivation of this general equation is given in the Second Report, together with expressions for the factors necessary for the application of the moment distribution method. These expressions will be found to agree with Professor Rathbun's factors in the particular case of the continuous beam (see the heading, "Moment Distribution Method"). It must be emphasized that in applying this method to a frame with semi-rigid connections it is essential to take into account the widths of the stanchions if serious errors are not to be made.[13] These methods of analysis are based on the assumption that the relation between moment transmitted by a connection and the relative rotation of the members joined, is linear. This is not true of actual connections and, therefore, in applying the methods an estimate, which may need subsequent correction, must first be made of the moment transmitted, as has been done in Table 3 of the paper. A direct determination of the restraining moments for such cases as those of Table 3, when a beam carries a uniformly distributed load, can be made by a simple graphical construction given by Professor Batho in the Second Report of the Steel Structures Research Committee, previously mentioned. If OPQ, Fig. 37, is the relation between angular rota-

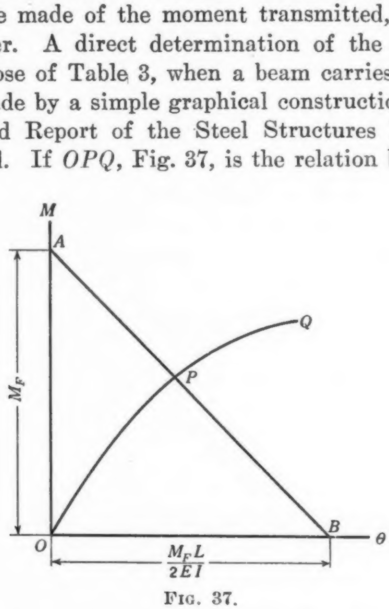


FIG. 37.

tion and moment for the connections, as determined by experiment, and if OA is measured off equal to M_F, the moment at the end of the beam which would cause complete fixity (that is, $\frac{WL}{12}$, in which W is the total load on the beam and L is its length), and OB is measured off equal to $\frac{M_F L}{2\,EI}$, and AB is joined, the point of intersection, P, of this line with the curve, OPQ, gives the end-restraining moment directly.

Mention has been made in Professor Rathbun's paper of the use of mechanical methods of stress analysis. Considerable attention has been given to this by the Steel Structures Research Committee. An attempt was made to use the deformeter on semi-rigid frames, the connection being obtained by reducing the rigidity of the beam as suggested by Professor Rathbun. This was found to be unsatisfactory due probably to the difficulty of making the necessary reduction with sufficient accuracy and also to the creep of the celluloid at the reduced section. A perfectly satisfactory direct loading method,[13] was evolved, however, in which the connections were made up in much the same manner as they are in practice.

L. E. GRINTER,[14] ASSOC. M. AM. SOC. C. E. (by letter).[14a]—It is encouraging that another writer has thrown the weight of his opinion, backed by the evidence of his tests, upon the side of those engineers who have continually insisted that riveted steel structures should be designed for partial continuity based upon the actual conditions of restraint. It is further encouraging that the author's tests present data that make it possible for one to estimate the probable restraint for a considerable range of beam sizes and types of riveted connections. Many more data of this kind are badly needed by the profession. In a country in which the annual construction program is several billion dollars, one has difficulty in understanding why the engineer must guess at such factors as end restraint when the cost of a comprehensive series of tests would be insignificant by comparison.

The tests presented are mainly on specimens of the connected beam type. Only Specimen 18 contained a column section between the beams. Hence, one must keep this fact in mind when using the author's data. It is evident that a girder-to-column connection will permit a greater total rotation than a girder-to-girder connection because of the duplication of tension rivets and the deformation of the column section. This effect can be evaluated for a specific case from the author's data. Specimens 18 and 15 have identical connections except for the insertion of the column section into the former. The moment-rotation curves, Figs. 21 and 23, show that Specimen 18 permitted an average rotation of about 20% more than Specimen 15. Of course, one could not expect this same relationship to hold for joints of considerably different types.

Of particular interest is the author's extension of the theory of continuous structures to take account of the elasticity of the connections. Much of the apparent complication of the procedure disappears when one has mastered the terminology involved. However, it seems to the writer that further simplification would be permissible without in any way destroying the true value of the analysis. For instance, consider the introduction of the terms, (a), (c), and (e), which greatly complicate Equations (17), (18), and (19). For all practical purposes the factor, (a), which is the distance from the face of the column connection to the first rivet through the beam flange, and the length,

[14] Prof. of Structural Eng., Agr. and Mech. Coll. of Texas, College Station, Tex.

[14a] Received by the Secretary March 25, 1935.

(e), which is the distance to the farthest rivet, might be taken as zero. Then, the factor, (c) = (e) − (a), would also be zero, and the effect upon Equation (18), for instance, would be as follows:

$$L_2 = L - 3\,e\left(1 - \frac{e}{L}\right) + \frac{b\,c}{2\,L^2}\,(3\,b + c) + 3\,E\,I\,Z\left(1 - \frac{a}{L}\right)$$

which becomes,

$$L_2 = L + 3\,E\,I\,Z \qquad (43)$$

when (e) = (c) = (a) = 0.

The physical significance of the suggested change is simply that the elastic action of the connection is assumed to be concentrated at the end of the beam instead of being distributed over the length of the end connection. Since there is still much to be learned about the true action of the connection itself, it seems quite as reasonable to the writer to assume that the elastic properties of the connection are thus concentrated. Undoubtedly, one could design special connections for which the assumption of zero length for the connection would be about as accurate as the author's procedure, which is based upon the assumption of uniform distribution of shear between rivets and a straight-line variation of beam moment within the length of the connection.

In order to demonstrate the insignificance of any error involved in the assumption of zero length for the connection, the writer has prepared Fig. 38, which is an analysis of the author's continuous beam example shown in Fig. 31 and analyzed in Fig. 33. The revised constants are determined, as follows:

$$L_{2BA} = 240 + \frac{3 \times 3 \times 10^7 \times 795.5}{10^8 \times 2.9} = 486$$

$$L_{2BC} = 120 + \frac{3 \times 3 \times 10^7 \times 215.8}{10^8 \times 4.2} = 166$$

$$A\,\overline{x}_{AB} = \frac{w\,L^3}{24}\;\frac{100 \times 20 \times 240^3}{24} = 1\;152 \times 10^6$$

$$A\,\overline{x}_{BC} = A\,\overline{x}_{CB} = \frac{100 \times 10 \times 120^3}{24} = 72 \times 10^6$$

$$M_{BA} = -\,\frac{3\,A\,\overline{x}}{L\,L_{2BA}} = -\,\frac{3 \times 1\;152 + 10^6}{240 \times 486} = -\,29\;500$$

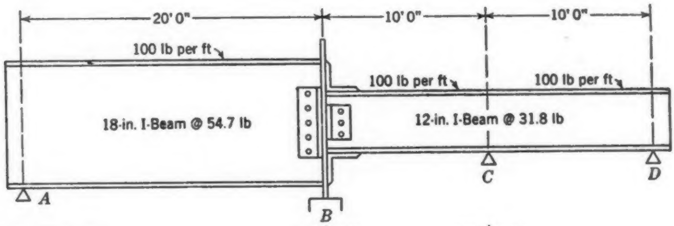
$$M_{BC} = +\,\frac{6\,A\,\overline{x}}{L\,(4\,L_2 - L)} = +\,\frac{6 \times 72 \times 10^6}{120\,(664 - 120)} = +\,6\;600$$

$$M_{CB} = -\,\frac{6\,A\,\overline{x}\,(2\,L_2 - L)}{L^2\,(4\,L_2 - L)} = -\,\frac{6 \times 72 \times 10^6 \times 212}{120^2 \times 544} = -\,11\;700$$

and,

$$M_{CD} = +\,\frac{3\,A\,\overline{x}}{L^2} = +\,15\;000$$

Signs agree with the writer's convention[15] that an end moment tending to rotate an adjacent joint clockwise is positive; the reversed moment is negative.



Stiffness Factors	1.63	1.59	2.19	1.80	
Distribution Factors	50.7%	49.3%	54.8%	45.2%	
Carry-over Factors	← 0	+0.5 →	← +0.361	0 →	
Fixed-End Moments	−29.5	+6.6	−11.7	+15.0	in.-lb ÷ 1000
Balancing Moments	+11.6	+11.3	+5.6		
		−1.8	−4.9	−4.0	
	+0.9	+0.9	+0.4		
			−0.2	−0.2	
Final Balanced Moments	−17.0	+17.0	−10.8	+10.8	

FIG. 38.—CONTINUOUS BEAM ANALYSIS FOR ELASTIC CONNECTION OF ZERO LENGTH AT JOINT *B*

The carry-over factor is different from + 0.5 for the left end of Beam *BC* where it is $\frac{L}{2\,L_2} = \frac{120}{2 \times 166} = 0.361$.

The stiffness factor for each beam is, as follows:

$$K_{BA} = \frac{I}{L^3} = \frac{795.5}{486} = 1.63$$

$$K_{BC} = \frac{4\,I}{4\,L_2 - L} = \frac{4 \times 215.8}{664 - 120} = 1.59$$

$$K_{CB} = \frac{4\,L_2\,I}{L\,(4\,L_2 - L)} = \frac{4 \times 166 \times 215.8}{120 \times 544} = 2.19$$

and,

$$K_{CD} = \frac{I}{L} = \frac{215.8}{120} = 1.8$$

A comparison of the results obtained by three analyses of this continuous beam is given in Table 4. It becomes evident from this study that the probable error involved in treating the elastic end connection as of zero length would be of no great importance in the design of an ordinary structure. The important fact (which is usually neglected) is that the elasticity of the end connection produces changes in moments of between 30 and 40% at both Point *B* and Point *C*. As compared with this large discrepancy, the small effect of the length of the end connection becomes of little importance.

It is interesting to speculate upon the difficulties involved in extending the author's treatment of the simple three-span beam involving only a single

[15] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 15.

elastic connection to a continuous frame with a dozen or more semi-elastic joints. Evidently, the labor involved would be a rather serious matter by any of the theoretical methods suggested by the author. In fact, a practical study would seem to be limited to the assumption that the moment-rotation curve remained a straight line or that the connection remained elastic within the limit of working stresses. The same limitation would necessarily apply

TABLE 4.—COMPARISON OF MOMENTS FOR THE CONTINUOUS BEAM ANALYZED IN FIG. 35.

Item No.	Assumptions	MOMENTS	
		M_B	M_C
	Moments, in Inch-Pounds, for:		
1..........	Inelastic connections............................	28 300	7 900
2..........	Elastic connections of zero length..................	17 000	10 800
3..........	Elastic connections by author's assumptions........	17 800	10 550
	Percentage Change from:		
4..........	Items Nos. 1 to 2..................................	−39.9	+36.7
	Items Nos. 1 to 3..................................	−37.1	+33.5
	Items Nos. 2 to 3..................................	+4.7	−2.3

to a deformeter study because it would clearly be impossible to represent the curved diagrams of the published tests by a stress-strain curve of celluloid. However, despite these apparent limitations, the author has performed a great service in calling attention to the serious effect upon moments and stresses that may be involved in the neglect of the elasticity and plasticity of the connections.

C. R. YOUNG,[16] M. AM. SOC. C. E., AND K. B. JACKSON,[17] ESQ. (by letter).[17a] —Within the scope of his tests the author has done a remarkably useful and convincing piece of work, and the rotations found for the connections tested can be used, no doubt, for estimating the probable behavior of similar details under certain conditions A more complete diagnosis of the cause of the rotations obtained, however, would have served the useful purpose of indicating the most effective remedies.

The design of the specimens raises certain questions in the minds of the writers who, some time ago, conducted a somewhat similar series of tests. The variety of types and sizes adopted by the author is most comprehensive, but it would have been desirable also to vary the length of the specimens in order to obtain the moment-shear relationship that exists in the simulated beam of length, $L = 12\ d$. Moreover, the column used in Specimen 18 was too short to develop its normal restraint on the connection.

Some difference of opinion may also arise as to the effect on the angular rotation of the omission of shear connections. The form assumed by the stems of the T's in flexure will not be the same for the top and bottom attachments. For the former the stem pulls away from the beam flange between the end of the beam and the first pair of rivets in it, whereas for the latter the

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[17a] Received by the Secretary April 1, 1935.

contact is firm from the end of the beam to the end of the connection. The curves of the two stems being different, additional rotation will naturally arise.

The method of measurement is also open to criticism. Stress distribution must be particularly irregular in all these specimens, and, therefore, the method of attaching the Ames dials and the location of their stem contacts are matters of first importance. In Series *B* and *C* (Fig. 6), the special device for holding Dials *E* to *L* required sleeve-jointed spring points; it was attached to the beam flange near the inner end of the connection, and the stem of the dial was kept in contact with a surface that moved transversely due to vertical shear. Can such mounting conditions warrant measurements to 0.0001 in.? Dials *M* to *P* were clamped to that section of the flange, just outside the connection, where buckling would first occur and the transverse movements of the dial stems on the plate due to flexure and shear might well excuse the inclusion of the results.

The writers are of the opinion that more reliable results are to be obtained by measuring the relative rotation of cross-sections of the beams beyond the local effects of the connections by means of flange extensometers that encircle the plate or column, and comparing them with similar measurements made on an uncut beam specimen similarly loaded. Intermediate extensometer measurements may be made to locate the sources of the rotation obtained, for example, elongation of the column flange tension rivets, flexure in the T, or deformation of the shear rivets in the beam flange. An independent check may be obtained by deflectometer measurements.

As the author has suggested, much additional information might have been obtained regarding the behavior of these connections from the systematic application of normal and reversed loads. Only by this means is it possible to dissociate the effects of flange attachment and stress on deformation and to detect the extent of elastic and non-elastic slip.

The phenomenon of greatly increased rotation on the first reversal of load for Specimen 14 is characteristic of riveted specimens. The reason for it is, of course, non-elastic slip. Had the reversal of loading been continued until the specimen reached a steady deformational state the rotation would have been the same for reversed load as for normal load.

It would be interesting to know to what relative extent the tension and compression attachments contributed to the rotation. The results of the writers' investigations indicate that the tension attachment is responsible for from 20% to 50% more deformation than the compression attachment. In seeking ways and means of improving end restraint, especial attention should be given, therefore, to the capacity of an attachment to resist tension with a minimum of yield.

University of Toronto Tests.—Incidental to an investigation of the degree of restraint developed at the ends of steel beams and girders with welded connections, a study was made of the rigidity of four riveted T-connections of two types at the University of Toronto, at Toronto, Ont., Canada, in 1930–32.[18]

[18] "The Relative Rigidity of Welded and Riveted Connections," by C. R. Young, M. Am. Soc. C. E., and K. B. Jackson, *Canadian Journal of Research*, July and August, 1934.

The connections were similar to those of the author's Series *C*, Specimens 14 and 15, except that shallow welded shear connections were added. They consisted of T's cut from 30-in., 149-lb, Bethlehem I-beams, connecting two 18-in., 47-lb. Carnegie beams to a central 1-in. plate for two duplicate specimens, and to the flanges of a 4-ft length of 12 by 12-in., 110-lb, Carnegie H-beam for two other duplicate specimens. For convenience, connections of the first type may be termed "plate specimens" and those of the second type, "column specimens." For the former, $\frac{7}{8}$-in. rivets were used throughout, but since the grip of the rivets in the flange of the T's was considerably less for the column specimens than for the plate specimens, they were made $\frac{3}{4}$ in. in diameter, the better to balance the design. This was in conformity with the expected strength of tension rivets of different grips, as indicated by tests made some years ago at the University of Toronto.[19]

A nominal factor of safety of between 2.25 and 2.50 was used in the design of rivets, selection of the working stresses being made with a view to rendering failure of the rivets more likely than failure elsewhere. Although the factor of safety adopted by the author for his Specimens 14 and 15 is not mentioned, it appears to have been greater than the foregoing.

Unlike the author, the writers distrusted the vertical shear rigidity of T's and, consequently, in order that the angular rotation of the connections at the face of the support might not be unduly influenced by vertical shear deformation, or that shear failure might not precede moment failure, two shear plates were welded on the web of each beam for all specimens, with a factor of safety of 3.4 in the welds.

In order to study the characteristics of the connections plain beam specimens corresponding to the cut-and-connected specimens were tested in exactly the same manner, and the effect of the connections was determined by differences.

Instruments.—Two types of instruments were used to measure the deformation of the specimens: (*a*) Yoke extensometers, with accessories, to measure horizontal deformation between points on or near the flanges of the beam; and (*b*) deflectometers, with accessories, to measure the vertical movement of points on the neutral axis of the beam.

Whereas the author measured his deflections with respect to a bar resting on rods fastened to the ends of the specimen, the writers measured the vertical movement of points on the neutral axis of the beams as related to a bridge hung from, and aligned at right angles to, the central vertical element of the specimen. Yokes carrying Ames dials were fixed to the specimens at the neutral axis and were of such length that the plungers of the dials bore on the bridge. The particular advantage of this method of measurement was that it gave the deformation of each half of the specimen with respect to the central vertical section separately, and, in the case of the connected specimens, indicated the relative rigidity of the two connected cantilevers.

[19] "Permissible Stresses on Rivets in Tension," by C. R. Young, M. Am. Soc. C. E., and W. B. Dunbar, *Bulletin No. 8*, Section No. 16, School of Eng. Research, Univ. of Toronto, 1928.

Test Procedure.—Although a loading in only one sense is required to determine the degree of restraint or continuity developed by a connection, a reversed loading is necessary for a wind-bracing connection, which may be subjected to complete reversal of stress, possibly involving both elastic and non-elastic slip. Furthermore, initial set occurring during the first load alters the behavior of the specimen during subsequent loads. Consequently, to determine the final rigidity of a wind connection, an adequate series of loads must be applied to eliminate initial set, and to reveal the full value of slip due to reversal of load, as well as the normal elastic deformation of the specimen. Inasmuch as the author's tests do not involve reversal of load (except in one incidental case) his rotation angles cannot apply strictly to wind connections after the first change in wind direction.

To fulfill the requirements mentioned each specimen was subjected to five loads applied and released in 5 000-lb. or in 10 000-lb. stages, from 0 to 75 000 lb, and a sixth load to failure. The first two loads, termed "normal loads," were applied so that the connection was stressed as by a gravity load in the simulated beam. The third and fourth, termed "reversed loads," were applied in the opposite direction, and the last two loads were again normal loads. Readings on the four extensometers and six deflectometers were made after each stage in the application and release of each load.

Results.—In the writers' investigation an effort was made to discover the sources of the deformation, an inquiry that the author did not attempt. For purposes of analysis the connections were divided longitudinally into three regions, I, II, and III, and transversely into two halves termed the top and bottom flanges. Region I comprises the half width of the column and, consequently, existed only for column specimens. Region II extended from the face of the plate or column $2\frac{1}{2}$ in. along the specimen, and Region III extended from the section at $2\frac{1}{2}$ in. to one at 12 in. from the face of the plate or column. For the plain beam specimens the origin was at mid-span. In referring to the specimens it was sometimes found convenient to group all three regions into one region—VI.

The flexural slope developed in a given flange and region is determind by: (1) The type of connection; (2) the flange included, top or bottom; (3) the stress involved, tension or compression; and (4) the loading sequence, including or excluding non-elastic slip.

In some types of connections the effect of a change in Factors (2), (3), or (4) is small, while in others it is considerable. A comprehensive test of any type of connection, therefore, must involve both flanges in both kinds of stress and the loading sequence must be such as to include any non-elastic slip that may occur. Consequently, the application of one-direction loads, as in the author's tests, can not yield the final answer to the problem of wind-bracing rigidity. It was for this reason that the writers adopted a load varying from capacity loading in one direction to capacity loading in the other direction.

Analysis of the flexural deformation occurring in each flange and region of the specimens under the reversing design load, combined with the evidence

obtained during failure loads, forms the basis of certain observations which attempt to explain the deformations indicated in terms of the structural details involved.

In Region I the most obvious deformational characteristic was that the tension half of the connection was the critical element. Elastic slip was slightly greater in compression than in tension, but was about twice as great in the top flange as in the bottom. Proportional deformation was 2.6 times as great in tension as in compression, but only slightly greater in the top flange than in the bottom. This excessive deformation in tension is due to the outward bending of the column flanges and the elongation of the connecting rivets.

The deformations in Region II included longitudinal deformation of the attachment within $2\frac{1}{2}$ in. of the column face, bending of the T-flanges, and deformation of the connecting rivets. The deformations of the top and bottom flanges were very similar in the riveted specimens, for which the top and bottom attachments were the same.

The deformations indicated for Region III included the longitudinal deformation of the attachment and beam, and the shear deformation of the connecting rivets or welds with any slip that might have occurred. Non-elastic slip produced more than one-half the deformation and constituted nearly one-third of the total deformation of this type of connection, elastic slip was negligible, and proportional deformation was similar to that of the plain beam.

An analysis of the deformation in Region VI disclosed the particularly interesting fact that for plate and column specimens the deformation arising in Region III is 75 and 50%, respectively, of the deformation in the entire connection. This substantiates Professor Rathbun's observation of marked deformation in the rivets through the beam flanges in the case of certain of his specimens.

Analysis of the total flexural deformation that occurred in each type during the reversing load and the relative effect of tension and compression, top and bottom flanges, and normal and reversed loads in producing it, justifies the following observations: (1) About 30 to 45% of the deformation consisted of non-elastic slip; (2) the aggregate elastic deformation in tension was from 56 to 70% greater than that in compression; (3) opposite flanges deformed equally; and (4) the deformations under normal and reversed loads were identical.

Coefficients of Restraint.—The coefficients of restraint (that is, the relation between the end moment developed in a uniformly loaded beam and the fixture moment), was first found for the initial load assumed as carried to failure. Account was thus taken of initial set, or slip, or both. Obviously, this is the correct loading for total moment calculations, as unrecoverable deformation in the connections permanently lessens their participation in the total moment developed in the span.

Although coefficients of restraint greater than unity are found for welded specimens, the values for riveted T-specimens were less than this. For the plate specimens, it ranged from 0.99 for 0.2 of the design load to 0.93 at

the design load. For the column specimens, it ran from 0.92 at 0.2 of the design load to 0.84 at the design load. Above the design load the values dropped off rapidly. Calculations show that for Professor Rathbun's Specimens 14 and 15 the coefficients of restraint at design loads were 0.96 and 0.73, respectively; for uniformly distributed gravity loads the ideal value, 0.75, permits an increase in load of 100 per cent.

A more typical situation with respect to end restraint arises when any one of the innumerable loadings following the initial load is applied. The connections have then suffered permanent deformation and permanent impairment of their restraining value. Calculations of the coefficient of restraint, therefore, should include a constant amount for the permanent deformation resulting between the beginning of the intial load and the beginning of the repeat load.

As was to be expected, the development of a large permanent deformation during the early loads reduced the coefficients of restraint under the repeat load in its early stages to values much less than those for the initial load. For the plate specimens, it varied from 0.63 at 0.2 of the design load to 0.9 at the design load; for the column specimens, it was much lower, varying from 0.3 at 0.2 of the design load to 0.75 at the design load.

Factors of Safety.—The plate specimens failed by flange buckling, with an average ratio of test factor of safety to design factor of safety of 0.97. The column specimens failed by rupture of the tension rivets, with an average ratio of test factor of safety to design factor of safety of 0.89. The rivets failed at an average tension of 26 900 lb each, or 61 000 lb per sq in., based on the size before driving, as compared with an assumed ultimate strength in pure tension of 30 000 lb. The fact that the column specimens gave smaller relative factors of safety than the plate specimens was due, no doubt, to a lesser uniformity of stress distribution among the tension rivets connecting the T to the column flange. Furthermore, the shorter grip of the smaller rivets in the column flange failed to compensate for their use in the column specimens and for the leverage effect of the pressure at the toes of the T. As the moments on the author's comparable specimens did not produce rivet tension in excess of 55 100 lb per sq in. (Specimen 15), the influence of flange bending and leverage on ultimate tensile strength was not disclosed.

Effective Slope Angles Due to Connections Only.—So far as the effect on positive moment in the beam or on story drift is concerned the slope angle due to the connection may be taken as the difference in flexural slope, at any section beyond the connection, between a beam integral with the column and a similar cut-and-connected beam under the same load.

For a load producing a moment varying from capacity positive moment to capacity negative moment, the slope angles attributable to the connections alone, were found by the writers to be 0.00222 radian for the plate specimens and 0.00492 radian for the column specimens. It is to be noted that in each case the shaft of the column is assumed to be straight, and that although local bending in the simulated column web is not included, bending in the column flange is included.

For a capacity moment in one sense only the values of the slope angles may be taken as one-half those given for the reversing load. The mean value for plate specimens under a load in one direction only (that is, 0.0011 radian) is approximately equal to that obtained by Professor Rathbun on reloading Specimen 15 at a moment of 40% of the ultimate moment, which was the ratio of design to probable ultimate load observed in the Toronto tests. It is not possible to make a similar comparison with the author's single column specimen, No. 18, as the ultimate moment is not given.

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DISCUSSIONS

HYDRAULIC LABORATORY RESULTS AND THEIR VERIFICATION IN NATURE

Discussion

By Messrs. W. F. Heavey, Chilton A. Wright, Paul S. Reinecke, Morrough P. O'Brien, and John A. Jameson, Jr.

W. F. Heavey,[4] M. Am. Soc. C. E. (by letter).[4a]—In 1933, an undistorted model of more than two miles of the St. Clair River, in Michigan, was constructed at the United States Waterways Experiment Station, which is the underlying subject of Lieut. Vogel's paper. The scale of the model was 1:100 and its purpose was to predict the behavior of submerged sills in the St. Clair River, which were proposed to raise the levels of Lakes Michigan and Huron in order to compensate for authorized diversions and for the enlargement of connecting channels. The series of model tests with different types of sills developed the conclusion that the desired compensation of 0.55 ft can be effected by as few as eight submerged sills, instead of the sixteen or seventeen originally estimated in 1926. To obtain maximum effectiveness from each sill it was found, from the model studies, that the up-stream face of each sill should be vertical, or nearly so, and that 60% of its effectiveness would be lost should material be deposited on the up-stream side of the sill to reach its top, which is 30 ft below low-water datum. Fortunately, the section of the St. Clair River at the outlet of Lake Huron carries very little material in suspension, and it is not expected that much fill will be so deposited.

Due to the peculiar difficulty of a mathematical solution to this problem and to the accurate and extensive data available for comparative purposes, the actual construction of these sills will afford an excellent opportunity to compare laboratory results with the action of Nature. The writer looks forward to successful vindication of the laboratory results, and hazards the prediction that not more than six sills will be required. Present plans are to construct only two sills as the first step and to observe the results obtained therefrom before constructing additional sills.

Note.—The paper by Herbert D. Vogel, Assoc. M. Am. Soc. C. E., was published in January, 1935, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[4] Major, Corps of Engrs., U. S. Army; Asst. Div. Engr., Great Lakes Div., U. S. War Dept., Cleveland, Ohio.

[4a] Received by the Secretary February 6, 1935.

In 1929 a new concrete caisson breakwater at Milwaukee, Wis., was damaged by a severe storm. At that time, there were several theories as to the reason for the failure. Apparently, it had failed because of scour on the harbor side, which had been thoroughly riprapped only a few weeks before the storm, but with smaller stones than had been used on the lake side.

A model of a small section of the breakwater on a scale of 1:24 was constructed in an indoor tank, 4 ft wide by 10 ft long by $3\frac{1}{2}$ ft deep. Small flux stone was crushed to scale and a device for creating waves corresponding to natural waves 20 ft high was installed. A cross-sectional view of the breakwater with rip-rap on both sides was available through a glass window in the side of the tank. As the size of the wave increased, it was observed that the drag of the receding wave and the waterfall action of the wave spilling over the breakwater were working stones loose on the harbor side, moving them back and forth until they were rolled some distance from the breakwater. After 60 hours of continuous wearing down of the rip-rap on the harbor side, the model breakwater suddenly failed. The larger sized rip-rap on the lake side was undamaged. This and subsequent experiments with larger rip-rap proved conclusively that, contrary to popular belief, heavy well-placed rip-rap is as essential on the harbor side as on the so-called "exposed" side of a breakwater in all cases where waves spill over the breakwater.

These examples are cited in support of the author's contention that model experiments may be profitably used in "verification" of the action of Nature, as well as in foretelling what the action of Nature will be on work not yet undertaken.

CHILTON A. WRIGHT,[5] ASSOC. M. AM. SOC. C. E. (by letter).[5a]—Field verification of the results obtained from model tests of hydraulic structures and rivers in several instances in which mobile sand beds were used in the models, is the basis of this excellent paper. Observations and surveys in the field yielded a good check as to the main features of the bed configuration and of the water stages developed in the model.

The present interest in the possibility of predicting from model tests the effects of control works on actual rivers impels the writer to cite an instance observed in Sweden in the summer of 1933. The regulating gates of the dam at the new low-head hydro-electric plant at Vargön are to be used to regulate the weekly flow of the Göta River at Lake Vänner, a natural lake about 2 140 sq miles in area.[6] The discharge of the river varies up to 30 000 cu ft per sec, the up-stream water level varies by 13 ft, and the down-stream level by 4.3 ft. These facts preclude the use of the usual discharge formulas.

The problem was taken to the Hydraulic Laboratory of the Royal Technical University, at Stockholm, where models of various types of sector gates were constructed in the hydraulic flume at a scale of 1:25, in order to determine

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[5a] Received by the Secretary January 18, 1934.

[6] "Untersuchungen betreffend die Abflussverhältnisse an Regulierwehr bei Vargön für die Wochenregulierung des Gota Älv," by Wolmar Fellenius and Erik G. W. Linquist. M. Am. Soc. C. E., Meddelande från Vattenbyggnadsinstitutionen vid Kungl. Tekniska Högskolan, Stockholm, Sweden, No. 7, June 1933.

the discharge as a function of the up-stream and down-stream depths and the gate-opening. Another model study of the entire project was made later at a scale of 1:36, in order to determine the interrelation of the discharges through the various gates and turbines.

The discharge diagrams obtained from the laboratory tests were compared with an official discharge curve developed previously for the Göta River from current meter measurements, and the diagrams showed a deviation of only −2% from the model results. A new set of discharge measurements was made at the weir in 1932 by means of a current meter, during a period of comparatively low water. No variation from the discharge curves based on the model tests could be observed over the range of these measurements.

This example corroborates the evidence given by Lieut. Vogel that in many cases the actual performance of hydraulic structures built according to designs determined by means of model studies in a laboratory can be predicted from these tests with a high degree of accuracy. Although the bed of the model was fixed in the case cited, rather than built of sand, as in the tests which Lieut. Vogel reports, the results obtained will be of equal interest.

PAUL S. REINECKE,[7] M. AM. SOC. C. E. (by letter).[7a]—The tests applied at the U. S. Waterways Experiment Station, at Vicksburg, Miss., to "prove" the correctness of a solution are described, clearly and concisely, in the paper. The "tests" are simple, understandable, and reasonable even to those who are not laboratory experts.

One reason why the solutions have been so uniformly satisfactory is that not only were visitors from the field invited to witness the tests, as explained by the author, but in the early three years of the Station's history (and the writer believes the custom is still being continued), it was obligatory for a field engineer, familiar with conditions at the site of the proposed improvements, to be present at the laboratory at least during the crucial periods of the test and preferably throughout the testing period. For this reason "theoretical" results were carefully "tied in" at all times with existing conditions and with practical results. Such a requirement prevented much waste motion in following a line of research which could not prove practical on account of field conditions that were not consonant with the theoretical assumptions made.

During the first few years after the establishment of the Waterways Experiment Station its personnel necessarily worked under a severe strain. Congress had adopted the Flood Control Plan which had to be set in motion promptly in order to avoid the danger of damage from another great flood. (So great was the early progress made that, in 1929, after less than one year's work, the repaired and improved Mississippi River levees withstood, without a single break, a flood equal to, or greater than, any previous flood in the river's leveed history, excepting only the Great Flood of 1927—a hitherto unprecedented record.) Consequently, it was necessary to have

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[7a] Received by the Secretary February 28, 1935.

reasonable answers for the details of "location" problems long before the laboratory could build up a well-trained staff or develop a thoroughly tested method of procedure. The only alternative was for the U. S. District Engineers in the Mississippi Valley to proceed along lines previously laid down and found reasonable, without any benefit of model study. In fact, it was necessary to proceed with considerable work before laboratory solutions were available and, naturally, in some few cases errors of judgment were made. For instance, training dikes were placed in several locations where satisfactory results were not obtained. Here, again, the Experiment Station came to the rescue of the country's pocketbook. Instead of scrapping these expensive completed works—costing more than $100 000, and building a new set elsewhere at similar expense—the laboratory solution indicated how a relatively slight modification of the existing system could be made to produce satisfactory river conditions. In this regard, it must be borne in mind that, in the present state of knowledge, the hydraulics—especially of stream flow—the construction, and the location of improvement works still involve more art than science, and, consequently, there may be more than one "satisfactory" manner of improving any certain reach of river.

With the assistance that model study was able to give, it was possible to avoid repeating the failures of approximately 50 yr ago in locating contraction or training works in the Lower Mississippi, where the great force and changing direction of the attack of the current during the varied changes of river conditions between high and low waters, exert such terrific strains on river structures. Due partly to their location and partly to their method of construction, the dikes and dams built in the Plum Point (Tenn.) Reach in 1881–85 were uniformly disappointing in permanent results obtained. The laboratory solution of the problems in this same reach bids fair to assist in improving permanently a troublesome stretch of river.

Model study, verified by the tests indicated by Lieut. Vogel, affords a reasonable answer, not only of the question as to where to build, but also of the question whether to build at all—as witness the problem at Morrison Towhead, below Cairo, Ill. Before the results of model study were available for this site, there was considerable argument whether the closing dike should be built at the head of the back channel or chute, at its foot, or near the middle. Many "pet ideas" were put forth, with the usual chance that the longest winded and loudest protagonist among all concerned would finally win. However, the results of trying out the different ideas in a model indicated quite apparently that the best answer was to let Nature alone at this place and save the Government's money.

In his remarks concerning studies on the Island No. 9 Model, Lieut. Vogel touches on the question of costs. The quoted figures indicate an average of about 30 000 cu yd dredged annually during the preceding 6 or 7 years. At a cost of 5 cts per yd, the annual charges amount to $1 500 per yr (disregarding expense of delays to navigation). The original plan of contraction works (8 500 lin ft at $35 per ft) would cost about $297 500, and the annual carrying charges at 5% for maintenance until the dikes have been silted in

permanently, and for interest, would be nearly $15 000 per yd—or ten times the cost of dredging. The approved solution of the laboratory, requiring only 1 600 lin ft of dikes, would cost only $1 600 to $1 700 annually. Consequently, in comparing the annual cost of the dikes with the cost of dredging, plus the expense to shipping on account of the delays, it is obvious that the improvement of the river by contraction and training works is warranted at this locality.

The foregoing analysis should help answer the question as to whether the building of contraction works in the Lower Mississippi is a waste of public funds, as some "dredging only" enthusiasts maintain. These latter do not in the writer's opinion give sufficient consideration to the extra expense caused by delays in transit while waiting for dredges to open up channels through numerous sand-bars developed after every high water. If it is possible, at reasonable cost, to contract and train the river to scour out these bars as the waters subside, and thus do away with the necessity of so many dredges, then by all means such training works should be constructed and the delays in transit due to shoal waters avoided. The laboratory affords a cheap and satisfactory method of estimating the efficiency and necessary length of training structures, and, consequently, their cost, to compare with the average cost of dredging in the same reach of river in its unimproved condition.

Very properly, Lieut. Vogel insists that any problem assigned for laboratory study must be assigned "without strings." A laboratory is a good and inexpensive place to try out "hobbies," provided it is not limited in experimentation only to those hobbies. Every possible and reasonable solution must be investigated, and the best ones recommended. In the early years of the Station (and also since as far as the writer knows personally) the Station Director was never limited to only one or two preconceived answers, but was urged and directed to get the best answer possible. The high type of engineering-scientific personnel on duty at the U. S. Waterways Experiment Station fully entitles them to the confidence placed in them. The author and his Staff are to be congratulated on the satisfactory and successful results obtained from their studies.

MORROUGH P. O'BRIEN,[8] ASSOC. M. AM. SOC. C. E. (by letter).[8a]—The importance of verifying the results of river model experiments by comparisons with Nature is emphasized in this paper. One criticism that may be offered is that the comparisons presented are restricted to river models, whereas the author implies in the "Introduction" that hydraulic models generally are being examined.

While he was a student in Germany, the writer was struck by the fact that, although great reliance was placed on model experiments of all kinds, few quantitative comparisons of models with Nature had been published. In "Hydraulic Laboratory Practice,"[9] which was a review of model theory and

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[8a] Received by the Secretary March 30, 1935.

[9] "Hydraulic Laboratory Practice," by John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., 1929.

results, almost no such verifications are to be found. In France, England, and the United States, the question of the reliability of models has been given more attention. An elaborate comparison of models of different sizes has been made in a paper by Carmichel and Escande.[10] This study covered the following subjects: Weirs, orifices, and tubes; uniform flow in open channels; transportation of steel spheres; surfaces of discontinuity behind obstructions; vortices, whirlpools, and gyrating motions; spillways and intakes; and flow in pipes at high velocities. In general, these experiments showed a scale effect that was not great. Carmichel and Escande also presented a complete and concise survey of model laws.

Concerning models of movable river beds and estuaries, Osborne Reynolds and Vernon-Harcourt felt the need for careful studies of the reliability of their results, but later experimenters have not always been as cautious. As to results obtained, Reynolds stated that, after a long period of operation, his model resembled the estuary of the Mersey River as well as charts, made at different times, resembled each other.

In the model of the Severn Estuary,[11] Professor A. H. Gibson investigated the discrepancy between results obtained with different vertical scales and different sands, and also compared his model with Nature. The scales finally adopted were 1:8 500 horizontally and 1:200 vertically, with a model sand size of 0.007 in. as compared with an average size of 0.009 in. Professor Gibson stated that these models reproduced, with a high degree of accuracy, the behavior of the tides at all points in the estuary, both as regards their height, range, and rate of rise and fall. Furthermore, the phenomenon of the bore was also reproduced with a very close agreement between the height and speed as measured in the model and as observed in the estuary. Observations of the drift of floats, and of current velocities at various points above and below the English Stones, he reported, agreed well with those at corresponding points in the estuary. Generally, all the phenomena connected with the movement of the water in the estuary were reproduced with a remarkable degree of accuracy in the model.

A comparison of the charts and cross-sections of the sand banks, silt deposits, and the main channels that were prepared from surveys of the model at times corresponding to 1886, 1901, and 1924, with corresponding charts and cross-sections of the estuary itself, showed a good general agreement over at least 90% of the area of the estuary. According to Professor Gibson, this agreement was as close as could be expected,

"In view of the fact that the model was not exposed to the effect of storms, which are liable to produce relatively large changes in any natural estuary. While there are differences in detail, these are not so great as the corresponding differences in detail in the estuary itself at the times of the different surveys. The only points of marked difference occur where, owing to some rapid deepening of the bed of the estuary, the slope of its sand banks is so great that the angle of repose of the sand in the model would have to be exceeded in order to give accurate reproduction."

[10] XVth International Congress of Applied Mechanics, Venice, 1931.

[11] "Construction and Operation of a Tidal Model of the Severn Estuary," H. M. Stationery Office, London, 1933.

The changes in the mean level of this estuary were reproduced with considerable accuracy in the model, and, generally, the agreement was bound to be such as to justify the assumption that "changes similar to those produced in the model by the introduction of a barrage would also be reproduced under similar conditions in the estuary."

For some time, the staff of the Hydraulic Laboratory at the University of California has been investigating the correspondence between models and prototypes. A few of the experimental results obtained by students have been published by the writer.[12] Additional work has been done by A. W. Kidder, Assoc. M. Am. Soc. C. E., the late John A. Jameson, Jr., Jun. Am. Soc. C. E., and others, and this work is being continued.

The discussion by Mr Jameson, published herewith, was prepared before his death to serve as his contribution to the data which Lieut. Vogel has offered in various papers. Its application to the present paper by Lieut. Vogel will be apparent from the content.

As would be expected, agreement between model and prototype is best when the reduction in size is not great and when the absolute depths and velocities considerably exceed certain critical values depending upon viscosity and surface tension. Furthermore, phenomena, such as flow over spillways, in which changes in pressure, momentum, and elevation are large as compared with the forces of friction, can be represented in models more faithfully than those in which friction is a dominant factor, as in models of rivers. In addition to the errors resulting from incorrect reproduction of the friction forces in fixed-bed models, river models having movable beds suffer from a lack of information on bed movement sufficient for the formulation of a proper model law. The author's comparisons of models and Nature show that in Nature there is a tendency toward the conditions found in the model, but on the basis of the data presented, one should not conclude that quantitative correspondence may be expected.

John A. Jameson, Jr.,[13] Jun. Am. Soc. C. E. (by letter).[13a]—For the past few months (1934) the writer has been engaged, at the Hydraulic Laboratory of the University of California, in a systematic study of the limits of hydraulic model correspondence, in the course of which certain observations have been made, which may be of interest.

True Models.—A model (scale 1:20) has been built of a concrete-lined tunnel conduit with its inlet and outlet transition connections to a concrete flume. In operating this model the elevation of the water surface in the flume below the outlet transition was controlled so as to correspond to that observed in tests[14] made on the prototype and the discharge for the model, Q_m, was

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[12] "Checks on the Model Law for Hydraulic Structures," *Transactions*, Am. Geophysical Union, 1932.

[13] Research Asst., Univ. of California, Berkeley, Calif. Mr. Jameson died September 26, 1934.

[13a] Received by the Secretary March 30, 1935, from Morrough P. O'Brien, Assoc. M. Am. Soc. C. E.

[14] Data presented to the Special Committee on Irrigation Hydraulics by Fred C. Scobey, M. Am. Soc. C. E. ((Unpublished).

adjusted according to the relation, $q = l^{\frac{5}{2}}$, or,

$$Q_m = \frac{Q_n}{1\,790} \quad \text{.........................} (1)$$

[in which q = the scale ratio of discharge, $\frac{Q_m}{Q_n}$; l = the scale ratio of length; Q_m = discharge through the model; and, Q_n = discharge through the prototype.]

Photographs of the prototype show a fish-tail wave in the outlet transition, which closely resembles that observed in the model. In Nature, fish-tail waves were observed at the outlet from the tunnel and a similar phenomenon occurred in the model.

Flow in the tunnel section of the model was characterized by standing waves. The first of these waves which occurred just below the tunnel entrance, had a height of about 15% of the water depth, while down stream the wave height decreased progressively. According to the observers who conducted the test on the prototype, no waves were observed when sighting through the tunnel, the flow being smooth. The possible reasons for their appearance in the model are: (1) The proximity of the depth of flow to the critical point; (2) the impossibility of obtaining correspondence of friction factors between the model and the prototype; and (3) the shortness of the approach channel.

Using the observed field data for the points at the ends of the tunnel, the value of Manning's N was found to be 0.0105. In a previous paper,[15] the author derives from the Manning formula the relation, $q = \frac{a^{\frac{5}{3}} s^{\frac{1}{2}}}{n\, p^{\frac{2}{3}}}$, or,

$$n = \frac{a^{\frac{5}{3}} s^{\frac{1}{2}}}{p^{\frac{2}{3}} q} \quad \text{.........................} (2)$$

[in which a = a scale ratio for areas, $\frac{A_m}{A_n}$; s = a scale ratio for slope, $\frac{S_m}{S_n}$; n = a scale ratio for roughness coefficient; and, p = a scale ratio for wetted perimeters, $\frac{P_m}{P_n}$.]

Since $a = l^2$, $s = 1$, $p = l$, and $q = l^{\frac{5}{2}}$, Equation (2) yields,

$$n = l^{\frac{1}{6}} \quad \text{.............................} (3)$$

For the model in question, $l = 20$ and $n = (20)^{\frac{1}{6}} = 1.65$, so that N_m for the model should be $\frac{0.0105}{1.65} = 0.00635$, in order to satisfy the foregoing relations. Actually, N_m was found to average about 0.0081 when computed from measured data for the same two points considered in the prototype computation.

The shortness of the approach section between the forebay and the model proper and the rather sudden transition from the forebay supply tank to the

[15] "River Laboratory Hydraulics," *Proceedings*, Am. Soc. C. E., November, 1933, p. 1420.

channel were necessary because of limitation of space. It was impossible to make the approach longer than about 18 in. (or about six times the depth of flow), and this condition resulted in undesirable turbulence in the flume approaching the tunnel. Several different types of baffles were investigated, but none seemed to have an appreciable effect on the standing waves in the tunnel, which may indicate that the latter were not caused by the shortness of the approach section.

When the results of the model tests are transferred to the prototype, the forebay level is about 10% higher in the model than in the prototype for the same discharge and down-stream water-surface elevation.

Distorted Models.—The distorted models included three open channels of rectangular section lined with painted galvanized iron, each consisting of two tangents connected by a 90° elliptical bend. These channels were the same depth (6 in.), but their horizontal dimensions were in the ratio of 1: 2: 4 (that is, considering the 12-in. channel as the prototype, the 6-in, and the 3-in. channels had horizontal scale ratios of 2 and 4, respectively). The bottoms of all three channels were horizontal and all opened into a large forebay tank at the upper end and discharged into a return channel at the lower end. In these tests each channel entrance was fitted with a galvanized-iron transition section to smooth out entrance eddies, and each had a galvanized-iron sharp-crested weir at both entrance and exit to isolate the section of the channel being studied from disturbance up stream or down stream. The entrance weirs operated in a submerged condition and their crests were fixed at the same elevation (about 2.5 in. above the channel bottoms). When the channels were calibrated individually it was found that the discharges were nearly in the ratio 1: 2: 4 (the quantity per foot of weir crest being: 12-in., 0.230 cu ft per sec; 6-in., 0.234 cu ft per sec; and, 3-in., 0.232 cu ft per sec. for a head on the weirs of 0.166 ft).

With water flowing through the channels sand was introduced into each immediately below the upper weir, the weights of sand being proportioned to the squares of the horizontal scales (that is, 1: 4: 16). When the water was shut off, and the channels were drained, it was observed that there was a definite line in the two larger channels above which there was no deposition whatever, due to the scouring action of the water below the entrance weirs. This line was parallel to that of the entrance weirs and not at proportional distances down stream as would be expected in scale models. In other words, the horizontal dimensions are controlled by the vertical, and distortion is not permissible. This condition also existed in model tests for the Hastings Dam, where a distorted model was used to study scour below gates.[16] The same line of scour would undoubtedly have appeared in the 3-in. channel if there had been longer tangent sections before the bends. As it was, the scour line for the two larger channels intersected the small channel in the curved part of its length so that no definite line appeared, although no sand remained in the up-stream tangent section.

[16] "Laboratory Tests on Hydraulic Models of the Hastings Dam," *Bulletin No. 2*, Univ. of Iowa Studies in Engineering.

Another factor that must be considered in using distorted models for experiments with bed load is the effect upon friction coefficients of the different proportions of bottom and sides making up the wetted perimeter in the model and prototype. In another series of tests, the entrance weirs were removed, and the lower weirs adjusted to compensate for the aforementioned friction differences. This was accomplished by raising the weir crests in the two smaller channels by computed amounts so that the slope of the water surface and the mean velocity would be approximately the same for all three channels. Wooden spur-dikes, 1 in. high, were placed radially at corresponding locations in the three channels, and the entire bottom of each channel was covered with about 0.4 in. of sand. The water passed through the channels at a depth of about 0.34 ft and at a velocity just sufficient for sand movement. The times of flow were in the ratio of 1:2:4. Although there was qualitative correspondence, it was observed that in spite of the fact that the weirs were adjusted carefully, so as to give equal mean velocities, the scour in the 3-in channel was more rapid than in the other channels.

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DISCUSSIONS

THE HYDRAULIC JUMP IN TERMS OF DYNAMIC SIMILARITY

Discussion

BY MESSRS. SHERMAN W. WOODWARD, ROBERT E. KENNEDY, L. STANDISH HALL, AND MORROUGH P. O'BRIEN

SHERMAN M. WOODWARD,[23] M. AM. SOC. C. E. (by letter).[23a]—After reading such an admirable exposition of a multiplicity of involved mathematical relationships, one feels a necessity for trying to reduce the complicated series of consecutive steps to some logical system simple enough to be pictured in one's mind as an entity. To do this it is necessary to review all the steps of the discussion and to attempt to isolate the fundamental connecting bases.

The entire mathematical theory for the hydraulic jump in a horizontal frictionless rectangular channel is based on two fundamental equations which, using the author's notation, are:

$$V_1\,d_1 = V_2\,d_2 \qquad (24)$$

and,

$$\frac{\nu\,V_1\,d_1}{g}\,(V_1 - V_2) = \frac{\nu}{2}\,(d_2^2 - d_1^2) \qquad (25)$$

Equation (24) is commonly designated in hydraulics as the "law of continuity"; Equation (25) is a change-of-momentum equation, a form of Newton's second law of motion. In Equations (24) and (25): ν = density of the liquid; g = acceleration of gravitation; V_1 = velocity before entering the jump; V_2 = velocity after leaving the jump; d_1 = depth before entering the jump; and, d_2 = depth after leaving the jump. Considering ν and g as known constants, the other four quantities remain as variables. From Equations (24) and (25) any one of the four variables, as may be desired, may be eliminated. If V_2 is eliminated, for example, the commonly used formula for the jump is obtained:

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2\,V_1^2\,d_1}{g}} \qquad (26)$$

NOTE.—The paper by Boris A. Bakhmeteff, M. Am. Soc. C. E., and Arthur E. Matzke, Jun. Am. Soc. C. E., was published in February, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1935 by Hunter Rouse, Esq.

[23] Prof. of Mechanics and Hydraulics, State Univ. of Iowa, Iowa City, Iowa; Cons. Engr., T.V.A., Knoxville, Tenn.

[23a] Received by the Secretary March 13, 1935.

In addition to the foregoing variables, the author introduces, among others, the following, all defined in terms of the original four:

$$q = V_1 d_1 \qquad (27)$$

$$d_j = d_2 - d_1 \qquad (28)$$

$$d_c = \sqrt[3]{\frac{q^2}{g}} \qquad (29)$$

$$\epsilon_{k1} = \frac{V^2_1}{2\,g} \qquad (30)$$

$$\epsilon_{k2} = \frac{V^2_2}{2g} \qquad (31)$$

$$\epsilon_1 = d_1 + \frac{V^2_1}{2\,g} \qquad (32)$$

$$\epsilon_2 = d_2 + \frac{V^2_2}{2g} \qquad (33)$$

and,

$$\epsilon_j = \epsilon_1 - \epsilon_2 \qquad (34)$$

With Equations (24), (25), and (27) to (34), inclusive, it is possible, theoretically, to eliminate any nine of the twelve variables, giving 222 different equations, each containing three variables. Each of these equations can be solved for each of the variables and, therefore, can be written in three different ways, making a total of 666 possible equations, any of which may be used if one should so desire. (Actually, the number would be slightly less, due to the fact that some of the equations would contain only two variables.)

Unfortunately, an equation containing three variables cannot usually be represented graphically by a single line, or in any other equally simple manner. It can be represented by a surface, however, or in a plane by a family of lines. To handle an equation of three variables, then, it is a common device to divide by one of the variables, or reduce the variables to ratios, in such a way as, in effect, to reduce one of them to unity, thus obtaining an equation which can be platted as a single line. With twelve variables, ignoring reciprocals, sixty-six different ratios can be obtained, all of which can be used to swell to still greater numbers the list of possible equations. It would even be possible to extend the combinations further by taking ratios of ratios; but it is to be remembered, in contemplating such an endless series of mathematical gymnastics, that the entire mass contains no more knowledge than the two original equations and the definitions of the variables.

It is not extremely laborious to plot a family of curves; several families can be put on the same diagram; therefore, a few diagrams would suffice to introduce all the variables desired. However advantageous it may be to develop a large number of equations for purposes of research, of theoretical

study, and of gaining an insight into the nature of the phenomenon, for the purposes of the practicing engineer, it seems desirable to adhere as closely as possible to the basic equations together with tables and diagrams to represent them.

ROBERT E. KENNEDY,[24] M. AM. SOC. C. E. (by letter).[24a]—It may be of interest to apply the data given in Table 1 to the formulas for length of the hydraulic jump suggested[25] by Donald P. Barnes, Jun. Am. Soc. C. E., who reports certain investigations made in Germany by Dr.-Ing. Kurt Safranez and quotes conclusions and formulas by Professor A. Ludin. The authors take issue with Dr. Safranez and contend that the jump is longer than that distance subtended by the visible roller as arbitrarily defined by him. Professor Ludin suggests an $\frac{L}{d_2}$-value of 4.5. The average of values in Column (4), Table 1, for the observed minimum length of jump divided by values in Column (3), for d_2, the down-stream water depth, happens to be exactly 4.5. The average for values in Column (5), the maximum length, divided by d_2 is 5.0 and ranges from 4.6 to 5.5 which is from 92 to 110% of the average. This is a span of 18 points. An $\frac{L}{d_2}$-curve is shown in Fig. 6.

Professor Ludin proposes a more accurate formula which may be derived from the following:

$$R = \frac{V_1}{V_c} \qquad (35)$$

and,

$$\frac{d_2}{L} = \frac{1}{4.5} - \frac{1}{6\,R} \qquad (36)$$

in which, in addition to the notation of the paper, R = a "flow index."[26] By combining Equations (35) and (36) and solving for L, the length of jump, a formula is obtained which, in the notation of Fig. 2, is:

$$L = \frac{9\,d_2\,V_1}{2\,V_1 - 1.5\,V_c} \qquad (37)$$

in which V_c is the critical velocity obtained from Equation (4).

The results of applying the data in Table 1 to Equation (37) are shown in Table 2. The observed maximum length varies from 55% of the computed for the lowest ratio of $\frac{d_2}{d_1}$, in Column (12), Table 1, to 86% for the highest ratio. If the percentages in Column (7), Table 2, are plotted against the corresponding values of $\frac{d_2}{d_1}$, two curves may be drawn to indicate the approxi-

[24] State Engr., Bismarck, N. Dak.

[24a] Received by the Secretary March 25, 1935.

[25] *Civil Engineering*, May, 1934, p. 262.

[26] *Loc. cit.*, Equations [1] and [5].

TABLE 2.—COMPUTED LENGTH OF JUMP COMPARED TO MAXIMUM OBSERVED LENGTH

Run No. (see Table 1)	Critical depth, d_c, in feet	Velocity, in Feet, per Second*: V_c	V_1	Length of Jump, L, in Feet: Computed	Observed†	Ratio: Column (6) / Column (5) (percentages)	Run No. (see Table 1)	Critical depth, d_c, in feet	Velocity, in Feet, per Second*: V_c	V_1	Length of Jump, L, in Feet: Computed	Observed†	Ratio: Column (6) / Column (5) (percentages)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
S 27	0.394	3.55	5.54	4.87	2.70	55.4	S 36	0.430	3.72	9.52	5.52	4.65	84.2
S 30	0.397	3.57	5.64	4.91	2.84	57.8	S 18	0.345	3.33	9.18	4.59	3.90	85.0
S 40	0.372	3.46	5.72	4.46	2.63	59.0	S 6	0.312	3.17	9.24	4.27	3.53	82.7
S 43	0.377	3.48	5.76	4.52	2.87	63.5	S 17	0.276	2.97	9.22	3.73	3.24	86.9
S 25	0.444	3.78	6.62	5.46	3.36	61.6	S 39	0.281	3.01	9.40	3.93	3.25	82.7
S 45	0.439	3.75	6.62	5.33	3.26	61.2	S 35	0.220	2.66	9.45	3.05	2.62	85.9
S 41	0.428	3.72	6.98	5.10	3.53	69.2	S 37	0.200	2.54	9.58	2.82	2.44	86.5
S 24	0.468	3.88	7.32	5.72	4.05	70.8	S 32	0.186	2.45	9.70	2.65	2.21	83.4
S 28	0.509	4.05	8.28	6.26	4.49	71.8	S 34	0.164	2.29	9.40	2.39	2.00	83.7
S 26	0.546	4.19	9.00	6.84	5.17	75.5	S 33	0.163	2.28	9.54	2.35	2.01	85.5
S 29	0.504	4.03	9.18	6.42	5.25	81.8	S 39	0.137	2.09	9.00	2.06	1.77	85.9

* See Fig. 2. † From Column (5), Table 1.

mate outside limits of the range of coefficients applicable to the Ludin formula and the discrepancy of this formula as based upon these data. Judging from these curves it appears to vary from 57 to 65%, or 8 points, for a $\frac{d_2}{d_1}$-ratio of 2.5 at the lower end of the curves and from 83 to 87%, or 4 points, for a $\frac{d_2}{d_1}$-ratio of 12 at the upper end.

For an illustration assume $d_1 = 0.36$ ft, $d_2 = 2.40$ ft, $V_1 = 17.2$ ft per sec, and $V_c = 5.7$ ft per sec. By the rough average approximation, $L = 5\,d_2$, the length is $5 \times 2.40 = 12.0$ ft. By Equation (27) $L = 14.4$ ft; or, applying the correction corresponding to a $\frac{d_2}{d_1}$-ratio of 6.7 which ranges from 82.5 to 86.5% and averages 84.5%, the length is equal to $14.4 \times 84.5\% = 12.2$ ft. In this case the "rough" approximation is not so rough!

L. STANDISH HALL,[28] ASSOC. M. AM. SOC. C. E. (by letter).[28a]—Designation of the component parts of the hydraulic jump in terms of a common parameter has reduced the characteristics of this phenomenon to a more readily understandable basis. The curves presented summarize the results of the experiments performed by the authors in a very concise manner. The differentiation between the use of the Froude and the Reynolds numbers is also notable.

With regard to the use of the results in engineering design, there are certain factors to be borne in mind which were not disclosed in the authors' experiments, which were made by producing a flow of high velocity by means of discharging water under a sluice. Flow of this type occurs from sluices

[28] Chf. Hydrographer, East Bay Municipal Utility Dist., Oakland, Calif.

[28a] Received by the Secretary March 29, 1935.

in canals or from sluices at the toes of dams. A notable example of the latter is to be found in the flood-control dams of the Miami Conservancy District, in Ohio. Hydraulic jumps also occur at the bottoms of chutes, either in canals or on spillways from reservoirs. They also form at the toes of overflow dams.

In all cases where the flow of water is allowed to accelerate over a sufficient distance, air is mixed with the stream, and the phenomenon of "white water" occurs. The same admixture of air with water can be observed in waterfalls, or in natural stream channels, when the velocity is sufficiently high to draw air into the water by its agitation. Very few data appear to be available as to the volume of air which will become mixed with the water under these conditions. Such data as are available indicate that it is in proportion to the velocity; in other words, a velocity of 10 ft per sec would entrain 10% of air by volume and a velocity of 40 ft per sec would entrain 40% of air by volume.

The entrained air increases the bulk of the water near the bottom of a chute, or the apparent value of d_1 is increased. When this mixture strikes the pool at the foot of a chute, the agitation in the pool is greatly increased. In an open pool, it is not probable that the depth, d_2, is changed from the value normally expected, as the air is rapidly expelled. However, the length of the jump is materially increased, and it is usually found in practice that the length of pool as determined by the ordinary hydraulic formulas is inadequate. Additional free-board is also necessary at the bottom of the chute and in the pool, due to the great agitation of the water before the air is expelled.

More data are required for the design of such structures, both with regard to the volume of air entrained at high velocities and also with regard to the effect of the air on the hydraulic jump. Data on the first phase of the problem are rather difficult to secure by laboratory methods. It would appear that the velocity, the distance traveled, and the roughness of the surface of the conduit had some bearing.

However, data on the effect of entrained air on the hydraulic jump could be easily determined in the laboratory. In the experimental flume used by the authors, air could have been introduced in the bottom down stream from the sluice and the effect on the jump noted. It is hoped that the authors will be able to expand their experiments to cover this phase of the hydraulic jump. In the meantime, designers should increase the ratio of $\frac{L}{d_2}$ to values greater than 5 when conditions are such that it is to be expected that air will be entrained in the water above the jump.

MORROUGH P. O'BRIEN,[27] ASSOC. M. AM. SOC. C. E. (by letter).[27a]—The preoccupation of engineers with the vertical elements of the hydraulic jump to the neglect of the horizontal dimensions is attributable to general amazement

[27] Associate Prof., Mech. Eng., Univ. of California, Berkeley, Calif.

[27a] Received by the Secretary March 30, 1935.

at the almost perfect agreement between theory and experiment as regards the depths and velocities involved. That so turbulent a phenomenon needed no "coefficient of ignorance" was both surprising and encouraging, and interest in it has helped greatly in advancing the application of the momentum principle to the solution of hydraulic problems.

One feature of the hydraulic jump which deserves attention is the fact that the equations apply to a frictionless liquid and yet a loss of energy is indicated. The theory provides no means of dissipating energy and one is left to assume that the energy at Point (2) in Fig. 2 is the same as at Point (1), but that a portion of it has become unavailable to cause further flow. The explanation appears to lie in the fact that although the sums of the pressure forces and dynamic reactions are equal and of opposite sign at the two points, the lines of application do not coincide and the fluid moving through the region of the jump is subjected to a continuously applied moment which superimposes a rotational velocity on the average velocity of forward motion. The energy involved in this rotational motion is the "loss" obtained from the combination of the momentum and energy equations. It should be emphasized that the mere existence of a surface "roller" could not cause this energy loss because little energy could be dissipated in the roller itself. However, if parts of this roller are continually torn away from it and pass down stream as vortices, the total energy lost is the summation of the energies of these vortices. In any real liquid, viscosity soon brings about the conversion of this macroscopic rotational energy into thermal energy. That the jump is not a reversible phenomenon in which the depth can decrease in the direction of flow from d_2 to d_1, which is possible from momentum considerations alone, is indicated by the impossibility of recovering this random thermal energy as energy of directed motion.

Another feature of the hydraulic jump which deserves more attention is the distribution of velocity in vertical sections, and the rapidity with which it approaches that resulting from friction alone. It does not appear to be safe to assume that, at the depth, d_2, the velocity distribution is normal for the reason that the highest velocities exist initially very near the bottom and there is no general criterion indicating the distance necessary for them to be reduced to a certain fraction of their initial intensity. It may be that a phenomenon occurs which is similar to "separation" along the blades of propeller pumps and turbines. Considering the layers immediately adjacent to the bottom, momentum and velocity must be lost at a rate sufficient to overcome both friction and the adverse pressure gradient and a point is reached at which the forward velocity is reduced to zero. The main stream may then separate from the bottom and pass over an eddy of sign opposite that of the surface roller.

In recent years, some difficulties have been encountered in the operation of hydraulic jump drops on irrigation canals. In these structures, the jump occurs in a pool at the bottom of an incline, and it may be that the usual pool length is not sufficient for the bottom velocities to be reduced to a safe value.

In most problems of open-channel flow, the energy and momentum curves can be computed from the average velocity (quantity divided by area), but where local conditions cause a considerable deviation from the normal velocity distribution, a correction factor must be applied to take into account the fact that the average of the momenta and energies transported across elementary areas exceed the momentum and energy based on average velocity.[28] The authors assume that this correction factor is unity.

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[28] "Velocity Head Correction for Hydraulic Flow," by M. P. O'Brien and J. W. Johnson, *Engineering News-Record*, August 16, 1934.

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DISCUSSIONS

WEIGHTS OF METAL IN STEEL TRUSSES

Discussion

By Messrs. Arthur M. Shaw, Joseph G. Shryock, Albert F. Reichmann, Robert W. Abbett, George C. Diehl, F. G. Jonah, Clarence D. Foight, A. H. Fuller, William E. Wilbur, W. N. Downey, J. R. Grant, Theron M. Ripley, and T. Kennard Thomson

Arthur M. Shaw,[8] M. Am. Soc. C. E. (by letter).[8a]—Methods for the quick determination of the weights of metal in bridge trusses are made available in this paper. The curves should effect a tremendous saving in time and effort which heretofore have been necessary in making preliminary studies of proposed structures. By the intelligent use of the graphs presented, it should be possible for the bridge designer to eliminate, promptly, several span-length combinations and types of trusses which, at first, might appear to be worthy of consideration.

Although the paper obviously is of primary importance to the bridge designer, the methods suggested may be used to great advantage by other engineers, especially those engaged in the location of railways and highways. In "easy" country, the locating engineer is not greatly concerned with the type of structure later to be adopted for crossing the various streams encountered. His main problems have to do with securing a reasonable alignment which will permit dropping by an easy gradient, down to an elevation not far above flood level and at a point resulting in the least possible length of bridge and approaches. Although he gathers data regarding flood flow, navigation demands, and (perhaps) makes something of a study of foundation conditions, in general, his activities and interests are quite fully divorced from those of the designing engineer of the bridge department.

The situation is far different when locating a railway or a highway in a more rugged region. The engineer must have available for convenient use in the field the means of determining, with reasonable accuracy, the cost of various alternate types of structures of varying span lengths, if he is to secure "the best location which the country affords."

Note.—The paper by J. A. L. Waddell, M. Am. Soc. C. E., was published in February, 1935, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion.

[8] Cons. Engr., New Orleans, La.

[8a] Received by the Secretary February 27, 1935.

As a concrete illustration of the foregoing, the writer recalls a railway location survey which was made in Northern Mexico about 1906. Within a distance of less than thirty miles, the projected line crossed three large arroyos. One of these arroyos was nearly a mile wide, with adjoining topography which permitted easy development to a low crossing but, for the other two, there was no alternative between high viaducts of considerable length and low crossings involving excessively heavy work and objectionable alignment.

The line was constructed as originally located, with a low crossing of the first arroyo (which was obviously proper) and high crossings at the other two, although the writer is not yet satisfied that the most economical location was secured. A serious effort was made to balance the costs of alternate plans, and for this purpose, the engineer was furnished with a certain amount of assistance in the form of assumed cost data, but these data were of doubtful authenticity and were neither sufficiently flexible nor comprehensive. Had the material then been available, which the author has presented, the problems could have been studied intelligently, and it is reasonable to assume that a more economical location would have been secured.

The locating engineer has neither the time nor the facilities for making a careful analysis of alternate structures at each stream crossing, and, in many instances, he would not be competent to make such an analysis even under favorable circumstances; but with all the aids with which he may now be supplied, he should be able to estimate weights and costs with a degree of accuracy which will enable him to follow the true principles of economics. He probably will not select the type of bridge and the arrangement of span lengths which, later, will be found to be the most economical, but his line will be located so that the most economical crossing can be secured. He will have a sound basis for his decision, instead of founding it on a "hunch," or on personal bias, as has been done too often.

JOSEPH G. SHRYOCK,[9] M. AM. SOC. C. E. (by letter).[9a]—The bridge engineer should find this paper valuable because it gives him at once the percentage ratio of the weight of metal per linear foot of truss in terms of the known total dead load and live load for all usual spans and types.

As a matter of interest, the writer made a check with the author's tables on an actual highway bridge. The roadway width was 20 ft; it was designed with the *K*-type of truss, for *H*-20 loading; its span was 330 ft; and the unit stress for the conventional reinforced concrete floor was 18 000 lb per sq in. Table 1(*b*) shows a ratio of 23%, which, adjusted for the unit stress intensity, gives 20.4%, whereas a detailed estimate gave a ratio of 20%, which is remarkably close—a variation of only 0.4 per cent.

In the last paragraph of his paper, the author makes a comparison on the Jacksonville Bridge of the saving of $22\frac{1}{2}\%$ in the trusses by the use of an open grating flooring in place of the ordinary reinforced concrete floor-

[9] Vice-Pres., Director, and Chf. Engr., Belmont Iron Works, Philadelphia, Pa

[9a] Received by the Secretary February 28, 1935.

slab. The writer made a comparative design and estimate, using a solid steel interlocking channel floor, with an asphalt plank wearing surface instead of the conventional concrete slab, on the aforementioned 330-ft highway bridge, which showed a saving of 22.6% in the carbon steel trusses, in addition to the saving of metal in the floor system. These combined savings more than offset the additional cost of the light-weight metal deck. As the spans increase, the actual savings in cost, of course, become more pronounced.

Some interesting bridges have recently been built by European engineers of chrome copper rustless steel, with physical properties comparable to silicon steel, and with a corrosion resistance four times that of the copper-bearing steel used in the United States. Recent developments in the metallurgy of steel and in the art of welding will doubtless play an important part in the design and construction of the great bridges of the future, and engineers in this country should watch with interest the remarkable progress being made in Europe.

ALBERT F. REICHMANN,[10] M. AM. SOC. C. E. (by letter).[10a]—Valuable data are offered in this paper, and the author has presented them in a convenient form which will enable a bridge computer or designer, readily and quickly, to secure the approximate weights of various types of trusses. This assembled information should be of great assistance in preparing preliminary estimates of cost and in determining preliminary dead loads. However, due to the many combinations of loading and the personal equation entering into a design, most engineers are reluctant to use weight curves to obtain dead load weights for final computations.

In the collection of curves presented with this paper, the author has omitted the continuous type of truss span. This type of span has gained wide recognition in the past decade (1925–35), and has proved economical for long-span bridges. Curves covering the weight of continuous spans might well be added to those of the cantilever and arch spans. With this addition, a designing engineer would be able to determine the approximate weights of the different types of spans, and thus be able to make preliminary estimates and comparisons of cost of various types of spans more readily.

Mr. Waddell is to be commended for contributing to the Engineering Profession a useful set of weight curves. They should prove of distinct value to engineers engaged in making preliminary studies of different bridge projects.

ROBERT W. ABBETT,[11] ASSOC. M. AM. SOC. C. E. (by letter).[11a]—The greatest value of the data presented in this paper will be in the field of highway bridge design. Heretofore, several writers have developed formulas for the weights of metal in such structures, but the results from them vary

[10] Asst. Chf. Engr., Am. Bridge Co., Chicago, Ill.
[10a] Received by the Secretary March 15, 1935.
[11] Asst. Prof. of Bldg. Constr., Union Coll., Schenectady, N. Y.
[11a] Received by the Secretary March 21, 1935.

over such a wide range that they are little better than guesswork. In general, the paper will be a distinct aid to younger engineers who have not yet developed the judgment required in preliminary bridge design.

Some pitfalls occur in the diagrams and their use should be tempered with reason. For example, the author states that,

"* * * they [the results from the graphs] are so accurate that, when properly modified for variations in the intensities of working tensile stress due to using design specifications other than those recommended by the writer,[2] it is probable that no recomputation will ever be necessary because of any serious discrepancy between the assumed and the calculated dead loads."

This statement implies one conclusion that is dangerous, namely, that if there is no serious discrepancy between the assumed and the calculated dead loads no further consideration is necessary. The writer would like to call attention to the fact that both the assumed and the calculated dead loads are considered as being uniformly distributed over the span. In bridges having spans in the longer range included by the author the actual distribution of weight may become an important design consideration. This is particularly true in the case of the cantilever and arch types and to a lesser degree in long-span, curved-chord, simple trusses. Recomputation should always be made on the basis of the actual distribution of weight in the case of large and important structures.

In cases where concentrated live loads are specified, the practice in determining an equivalent uniform load is not standardized. The author recommends the following method for railroad truss bridges:[12]

"If one will figure on cars preceding as well as following the locomotives and will compute the maximum moments at the quarter points of the spans, then substitute them for M in the proper formula ($M = \frac{3}{32}\, w\, l^2$), he will obtain the best averages for the equivalent load curves."

Presumably, the same procedure would be utilized for highway bridges in determining a uniform live load to be used with the diagrams.

Suspension bridges and bascule spans have been excluded from the scope of the investigation, and the author is correct in stating that it is not feasible to plot these types. Recognizing this fact, the diagrams provide a quick method for the preliminary design of all types of steel bridges with the exception of the simple-span, steel beam, the rigid frame, and the continuous truss.

The first two of these types are used for short spans and to estimate their weights is not an outstanding difficulty in design. The continuous truss, on the other hand, is becoming such an important type that it promises to replace the series of simple spans almost entirely, and perhaps the cantilever as well. If this is true there is a definite need for a diagram of weights for the continuous bridge similar to those presented in this paper.

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[2] "Bridge Engineering," by J. A. L. Waddell, M. Am. Soc. C. E., Vol. 1, Chapter XIV and LXVIII, John Wiley & Sons, New York, 1916.

[12] "Bridge Engineering," by J. A. L. Waddell, M. Am. Soc. C. E., Vol I, p. 166.

GEORGE C. DIEHL,[13] M. AM. SOC. C. E. (by letter).[13a]—The bridge engineer who has not had extensive experience and training is enabled, easily and quickly, to determine the most economical form of steel bridge construction, by the aid of this paper. Table 2 is especially interesting in showing the great accuracy of the charts which the author has prepared. The last paragraph of the paper is especially engaging, as it indicates the possibility of lighter floorings on highway bridges and shows how the open-grating floor will save in the trusses alone, one-fifth the cost if carbon steel is used, and one-sixth in case of silicon steel. Similar charts should be prepared which would show with equal facility and clearness the lessened cost of floor systems if the lighter floors were used and also the relative cost of the different types of floors.

In the United States the perils to life and limb, as well as the property damage on the public highways are still on the increase. More than 36 000 deaths occurred in 1934, and nearly 1 000 000 people were injured. Many of these hazards occur by skidding on bridges. It is important, of course, that a bridge should be economical in its cost, but it is even more important that it should be safe horizontally as well as vertically. Safety from automobile accidents can be secured at lower cost. This applies particularly to new bridges. The lighter weight of open flooring makes its use practicable on existing structures, when it is found necessary to eliminate slippery roadways and to prevent accidents caused by ice and snow.

A valuable addition to the paper, would be a curve of relative cost of bridges with the conventional types of solid floor as contrasted with open, non-skid flooring.

F. G. JONAH,[14] M. AM. SOC. C. E. (by letter).[14a]—The paper offers in a condensed, workable form, a set of curves platted from information accumulated during years of practice and experience. The author is to be commended for his work of assembling and classifying these data and passing it on in a condensed graphical form to the Engineering Profession.

Figs. 1 to 5 will be of great value to bridge engineers (especially to those of less experience) in making preliminary studies and selecting economical types of design for proposed bridges. It will give bridge estimators reliable information that has not been available in such concise and workable form. The percentage-ratio weights of a number of the more recent carbon-steel, simple-truss railway trusses constructed on the Frisco Lines have been "spot"-checked and found to agree very closely with the curves of this paper.

CLARENCE D. FOIGHT,[15] M. AM. SOC. C. E. (by letter).[15a]—There is no doubt but that the author has made a valuable contribution to bridge designers. In using his curves, however, the designer should know the height ratio, the

[13] Cons. Engr., New York, N. Y.

[13a] Received by the Secretary March 25, 1935.

[14] Chf. Engr., Frisco Lines, St. Louis, Mo.

[14a] Received by the Secretary March 27, 1935.

[15] Designing Engr., Gulf Refining Co., Pittsburgh, Pa.; formerly Designing Engr., Div. of Bridges, Bureau of Eng., Pittsburgh, Pa.

[15a] Received by the Secretary March 15, 1935.

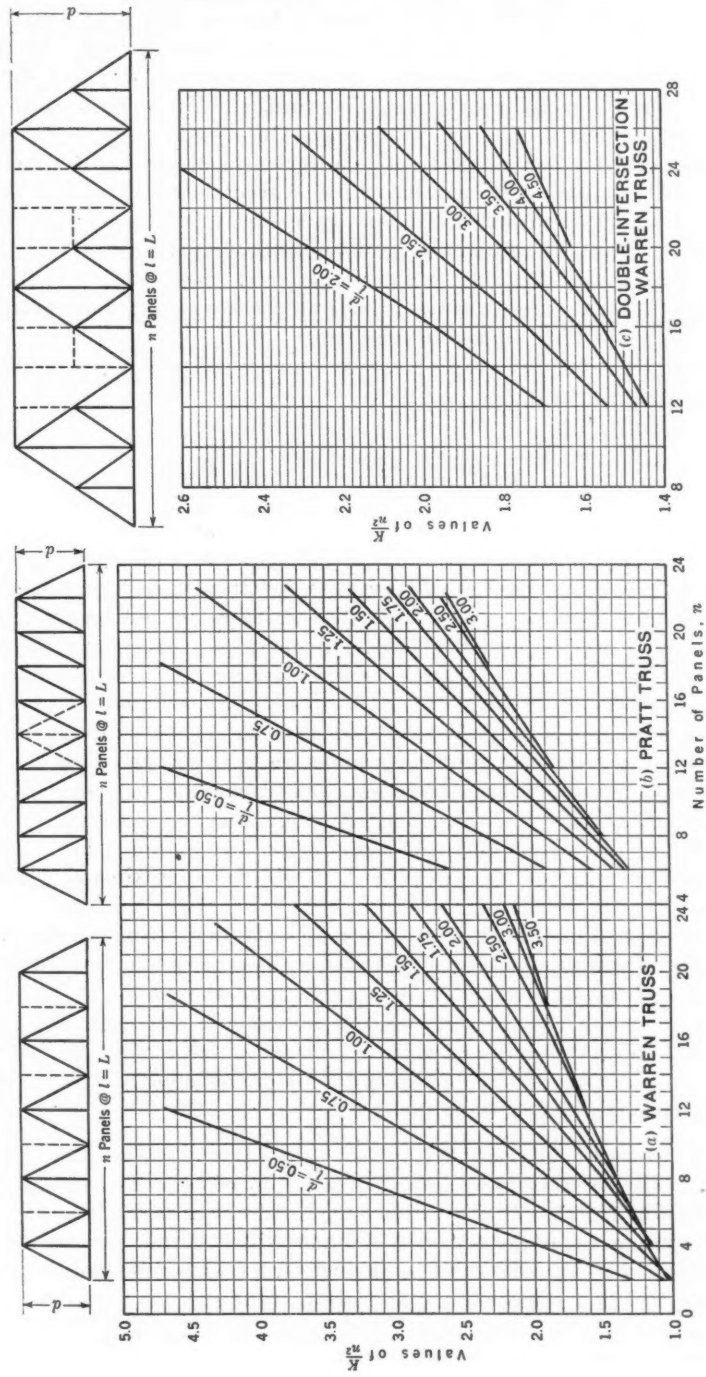

FIG. 7.—COEFFICIENT CURVES.

number of panels, and the type of diagonal bracing that have been used in the design of the bridges from which the curves were derived. Should the most economic height ratio for any span be known, there will be a difference in truss weight for different types of trusses, and for trusses of the same type having different numbers of panels. On a number of occasions, the writer has compared different types and variously proportioned simple span trusses and has found that a comparison of several trusses having different numbers of panels and different height proportions is generally required before one is able to determine the most economic proportions.

Fig. 7 is a tool for such comparisons which may help to explain discrepancies in truss weights derived from the author's empirical curves when the truss is not a truly economic structure.

The curves, applying to three common simple-span truss types, are referred to the rational formula,

$$w' = \frac{w_c \times 3.4\,(1 + p)\,L\,\dfrac{K}{n^2}}{f - 3.4\,(1 + p)\,L\,\dfrac{K}{n^2}} \qquad \text{(3)}$$

in which w' = average weight of metal per linear foot of truss (not including the members in Fig. 7 which are shown dotted); w_c = average weight per linear foot, carried by truss (includes floor system, all bracing, railing, uniform live load, uniform live load equivalent to a concentration of load, and impact); p = ratio of details to gross section; f = average allowable fiber stress in gross section; and L = span length. Equation (3) can be relied upon to give the truss weight to an accuracy of within 5 per cent.

The proportion of details to the gross section, p, will generally vary from 0.30 to 0.35 for pin-connected trusses, and from 0.35 to 0.40 for riveted trusses. The average allowable fiber stress in the gross section, f, will be about 75% of the allowable tensile stress in the net section; that is, when a stress of 16 000 lb per sq in. is used, f will be approximately 12 000 lb per sq in.

Reference to Fig. 7 will demonstrate that the most economic height proportion varies with the number of panels and the type of bracing. In Table 2, the weight of the Springfield, Mass., bridge from the curves is 15% lighter than as computed. In the case of the Portsmouth lift-span, the error is 8 per cent. Since the author has not given the number of panels, the depth of truss, or the type of bracing to which his curves apply, large errors may result from variations in these factors.

A. H. Fuller,[16] M. Am. Soc. C. E. (by letter).[16a]—The data given in this paper will be of inspiration and of value in office and classroom. Their usefulness will increase as the various results are compared with other available information. The experienced designer will have his own weight data, and will find this paper of value as it touches a field beyond the limit of his experience and as a comparison with his own results.

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[16] Prof., Civ. Eng., and Head of Dept., Iowa State Coll., Ames, Iowa.

[16a] Received by the Secretary March 28, 1935.

The young designer and the student who is studying bridge design will find in the paper a marvelous organization of desirable information which has not previously been available. Such men, in considering weight data, should keep in mind the following points: (1) That only a trained designer, working with known specifications, can expect to approach precision in his estimates; (2) that data are now available by which the man with limited experience may make quite a satisfactory preliminary estimate of weight of various types of structures for any loading and specification; (3) that, for those with limited experience in estimating details, the preliminary estimates are likely to be as close to actual weights as computations based upon main sections only; and (4) that weight data, in proper form, are useful in determining the preliminary economic layout of spans for multiple-span bridges.

In 1932 the writer, with Frank Kerekes, M. Am. Soc. C. E., prepared an article[17] on dead loads for simple truss highway and railway bridges. In order to meet the double requirement of making preliminary estimates of weight and to have a basis for the determination of economic span lengths of simple span bridges he chose the often used expression:

$$w = CL + F \qquad (4)$$

in which (in the author's notation), w = weight of steel, in pounds per linear foot of bridge; L = span length, in feet; and C and F are constants. Values of $C = 4$ and $F = 500$ were used for through, riveted, highway bridges with a 20-ft roadway and 8-in. concrete floor-slab on steel stringers, and for the H-15 loading. Values of $C = 10$ and $F = 1\,200$ were adopted for single-track, through, riveted, railway bridges, with E-60 loading and ordinary open floor. The spans in all cases were limited to lengths from 100 to 300 ft. Suggestions were given for modifying the constants to provide for variations in thickness and type of floor, width of roadway, and live load.

The writer has been much interested in making comparisons between results as given by Equation (4) and as taken from Figs. 1 to 5 of the paper. The author's total weights, as taken from Fig. 3 for Class B highway bridges of carbon steel, are heavier by 30%, or more, than the structures upon which the writer based the constants for his formula. By about the same percentage the author's Class A bridges are heavier than Equation (4), with its proposed reduction factor, would suggest for H-20 loading. The railroad bridges cited in the paper are lighter than the writer's examples by about 10 per cent. The writer believes that Equation (4), with proper constants, is more flexible than the curves in Fig. 3 for making a preliminary estimate of weight.

Differences in loads, specifications, and personal equations in designers, are reflected in the differences in weights between those given by the author's curves and those compiled by the writer. Perhaps no appreciable good would result in an attempt to reconcile them. The author states that he gives the actual weights only as a start for using the percentage data in Figs. 1 and 2. These percentage data, with supporting discussion, form the true basis for

[17] "Analysis and Design of Steel Structures," by A. H. Fuller and Frank Kerekes, D. Van Nostrand Company, Inc., New York, 1933.

the paper. The writer finds a much closer comparison with the percentage data than with specific weights. In fact, he would need a definite separation of truss and bracing weights (which are available only in a few instances) to determine whether his percentages were higher or lower than those given by the author. In Equation (4), the term, $C\,L$, includes the trusses and that part of the bracing which varies with the span length; and the factor, F, includes the stringers, floor-beams, and the part of the bracing that is independent of the span.

Using the product, $C\,L$, as representing the weight of trusses for highway bridges, the writer derives percentages of truss load to total load which are uniformly about 10% higher than those of the author. This 10% is a fair value for the part of the bracing that varies with the span length. The check, then, is remarkably close for highway bridges. For railroad bridges, the writer found a greater discrepancy, but in the same direction. The effect of the bracing will reduce the differences to negligible magnitudes.

The writer accepts the percentages of the paper as a definite contribution to the problem of making and adjusting estimates for weights of bridges. The extension of data from simple spans of moderate length to simple spans of great lengths and to cantilever and arch bridges makes this paper a notable addition to the literature on this subject.

WILLIAM E. WILBUR,[18] M. AM. SOC. C. E. (by letter).[18a]—For some years the writer has been using curves presented by the author,[2] or others similarly constructed, for preliminary estimates, as well as for obtaining an assumed dead load for designing. This paper which brings the information up to date, constitutes a welcome addition to the available tools of the designing engineer.

The device of expressing the weight of the truss as a percentage of the total load results in a considerable simplification of the diagrams. The process could readily have been carried one step further, by expressing the truss weight as a percentage of the superimposed load; that is, the total load exclusive of the weight of truss. When this is done, the weight of the truss is obtained directly, without the necessity of first assuming its weight before entering the diagram, as was done in the example given by the author. In Fig. 8, the author's diagrams, Fig. 1(*c*) and Fig. 1(*d*), giving weights of simple trusses of carbon and silicon steel, have been replotted on this basis.

The weights given in the paper agree quite closely with the experience of the writer. Assuming the same basic unit stresses, the minor differences of specifications will have no great effect on the weights of properly designed trusses. For ordinary highway bridges of carbon steel and moderate span, with a basic unit stress of 16 000 lb per sq in., the writer has found the following formula quite satisfactory:

$$w = \frac{1}{2}\,L + \frac{w_t\,L}{1\,600} \quad \text{.....................(5)}$$

[18] Asst. Engr., Harrington & Cortelyou, Kansas City, Mo.

[18a] Received by the Secretary April 10, 1935.

[2] "Bridge Engineering," by J. A. L. Waddell, M. Am. Soc. C. E., Vol. I, Chapters XIV and LXVIII, John Wiley & Son, New York, N. Y., 1916.

in which (conforming to the notation of the paper), w = weight of truss, in pounds per linear foot; L = span length, in feet; and w_t = load on one truss, including weight of truss, in pounds per linear foot. In terms of the

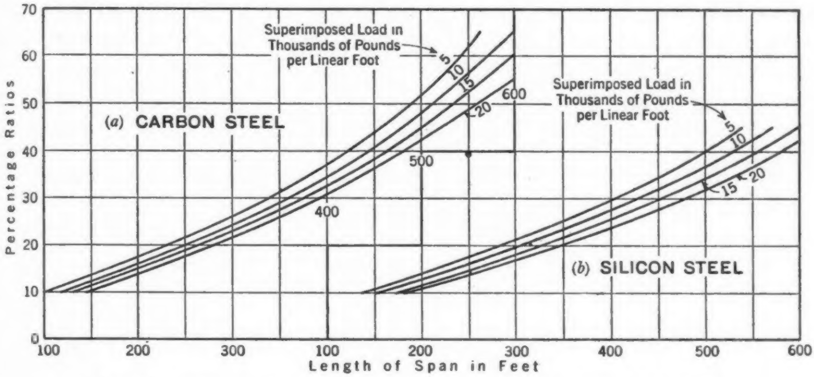

FIG. 8.—PERCENTAGE RATIOS OF WEIGHTS OF METAL FOR SIMPLE BRIDGE TRUSSES IN GENERAL.

plotting used by the author (Fig. 1(c)), this gives a straight-line variation. The author's lines are slightly curved, which is more accurate for the longer spans and heavier loadings.

Attention should be called to the fact that the loads per foot, given in the author's diagrams, are the total for two trusses, rather than for one. The minimum given, 10 000 lb per ft, is rather too heavy for short, light spans. For highway bridges of 20-ft roadway with concrete floor, designed by specifications for H-15 loading, advanced by the American Association of State Highway Officials, the total load per foot of span for short spans is only slightly more than 5 000 lb.

In his previous paper on "Economic Proportions and Weights of Modern Highway Cantilever Bridges."[19] the author gave, for structures of combined silicon and carbon steel, an estimate of the percentage of each kind of steel in the structure. This information is of great value in making preliminary estimates; and the writer would suggest that the value of the paper will be considerably increased if the author will include, in his closing discussion, such data for the structures covered in this paper.

W. N. DOWNEY,[20] Assoc. M. Am. Soc. C. E. (by letter).[20a]—The Engineering Profession is indebted to the author for considerable information concerning the weights of metal in bridges; his paper is a notable contribution, covering as it does such a variety of types. The following discussion is restricted to comments on the author's weights of simple-span highway bridges. The curves submitted for comparison with Figs. 1 to 5 were designed

[19] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 888.

[20] Care, State Highway Dept., Frankfort, Ky.

[20a] Received by the Secretary April 22, 1935.

to carry live loads and impact loads in accordance with the recommendations of the American Association of State Highway Officials (1931). The allowable unit stress in tension was taken at 16 000 lb per sq in. on the net section except where noted. The designs provided for a concrete floor and a future surface item, the combined weight of which was 115 lb per sq ft of floor area.

The writer obtained a close check on the percentage ratios of Fig. 1(*b*), his values being about 1% higher than the ratios shown by the author. This is gratifying since it gives a check on the portion of the bridge weight that is most difficult to estimate, namely, the trusses. In this connection, Fig. 9

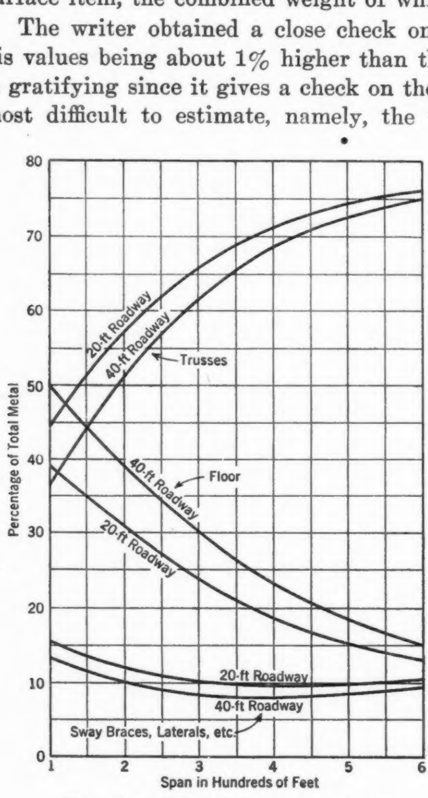

FIG. 9.— DISTRIBUTION OF METAL.

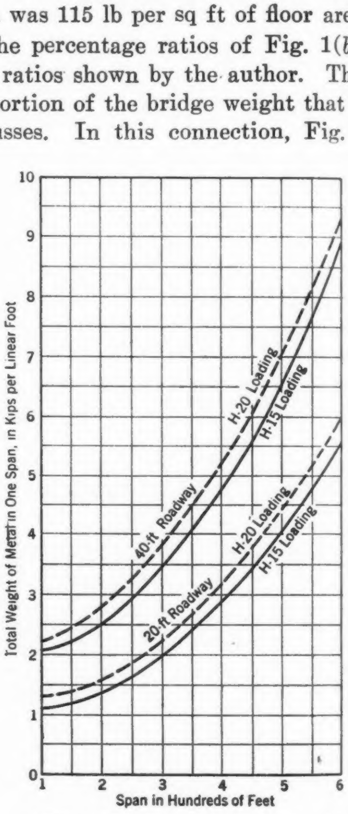

FIG. 10.—TOTAL WEIGHT OF METAL. (1 KIP = 1 000 POUNDS.)

shows the distribution of the metal between the major elements of the span. The weights of hand-rails, expansion devices, and shoes are not included in the weights given in the paper.

Presumably, the total weights of metal given by Fig. 3 (*b*) include the weights of the hand-rails and expansion dams. The total weights obtained by the writer for bridges with 20-ft roadway, are in good agreement with Fig. 3(*b*) for the longer spans, but he gets somewhat smaller weights for spans of less than 300 ft.

The weights in Fig. 3(*b*) for bridges with 45-ft roadway appear to be much too great. Examination of this set of curves shows it to be based on the assumption that the weight of metal per square foot of roadway is constant for all widths of roadway. This is not in accordance with the facts;

a certain portion of the structural steel in a span is independent of the roadway width. From a study of data at hand the relation expressed by Equation (6) was found to give good results. The weight of metal may be determined for any wider roadway from the weight for a 20-ft roadway by the following equations:

$$w_w = C\, w_{20} \qquad \text{(6)}$$

in which w_w = weight of structural steel, in pounds per square foot of roadway for the wider roadway; w_{20} = weight of structural steel, in pounds per square foot of roadway for the 20-ft roadway; and,

$$C = 1 - \frac{R'}{100}\left[0.10 + \frac{R'}{100} \times \frac{L}{500}\right] \qquad \text{(7)}$$

in which R' = the width, in feet, of the wider roadway; and L = the span, in feet, of the bridge.

Attention is called to the fact that, in Equation (6), w_{20} must be the weight of metal, in pounds per square foot of roadway for a bridge with a 20-ft roadway. Suppose, for instance, that for a certain span, the weight of metal is known for a bridge with a 30-ft roadway and it is required to find the weight for a bridge having a 40-ft roadway. First, using Equations (6) and (7) and the known weight, w_w, for the 30-ft roadway, compute the weight, w_{20}, for a 20-ft roadway bridge. Then, with the value of w_{20} known,

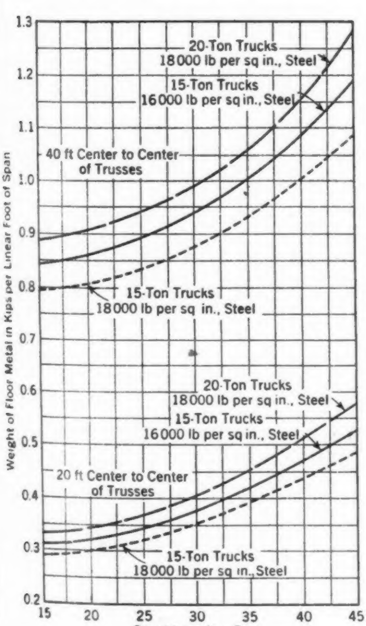

FIG. 11.—WEIGHT OF METAL IN FLOOR FOR ONE SPAN.

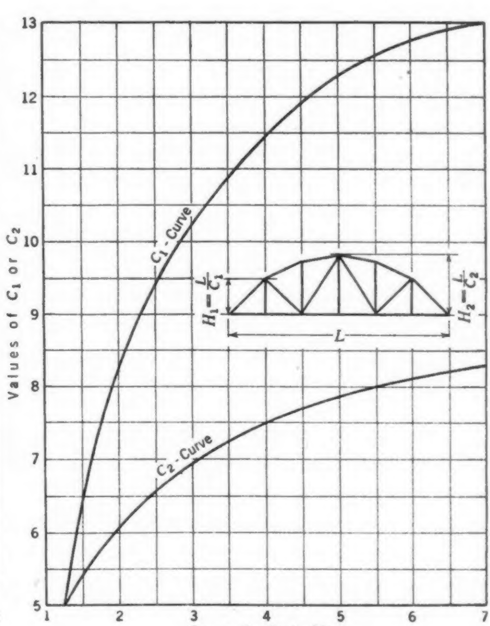

FIG. 12.—DEPTHS OF TRUSSES WITH POLYGONAL TOP CHORDS AND 20-FOOT ROADWAY.

apply Equations (6) and (7) again to find the weight of metal, in pounds per square foot, in the bridge with a 40-ft roadway.

Fig. 10 shows the total weights of metal in the spans as determined by the writer and is included for comparison with the values given in Fig. 3(*b*). Fig. 11 permits the determination of desirable values that are not directly determinable by Figs. 1 to 5 of the paper. It yields the weights of the floor system and, incidentally, shows the effect on the floor weight of changing the panel length.

As is demonstrated in the paper, the depth of truss affects its weight materially. Coefficients for the determination of the truss depths for bridges with 20-ft roadways may be read from Fig. 12. For roadways wider than 20 ft add 1 ft to the center depth for each 5 ft of additional roadway width. Trusses of these proportions have been found to be pleasing in appearance and economical of metal.

J. R. GRANT,[21] M. AM. SOC. C. E. (by letter).[21a]—The Engineering Profession is deeply indebted to the author for making available to bridge engineers the curves of truss weights and other valuable information included in this excellent paper. It is essential, particularly during the early stages of a bridge project, to have reliable approximate weights of the steel part of the structure on which to base estimates of the cost of alternative plans. The floor can be computed quickly and the bracing can be estimated, so that with the truss weights readily available from the curves, an approximate estimate of the weight of steel in the superstructure is soon obtained.

The writer checked some of the percentages given by Figs. 1 to 5 for a number of spans for which he had the weights available, and although there was considerable variation in some cases it was usually not difficult to account for the difference between the computed percentage ratio and that from the curves. They should enable any bridge engineer, who is familiar with the design of the class of structure proposed, to make a fairly close estimate of the truss weights. It is essential that due allowance be made for the various factors, besides unit stresses and variation from the economic depth, which will have an influence on these weights. In considering combined railway and highway structures, the writer found that the percentage ratio from the curves for the railway part of the bridge, when increased or decreased by one-third the difference between this percentage and that for the highway part, checked very well with the computed or actual weight percentage.

THERON M. RIPLEY,[22] M. AM. SOC. C. E. (by letter).[22a]—In this paper the author has added another rung to the ladder by which the general practitioner can rise to an intelligent answer to many questions without spending hours in detailed computations to secure a general result.

An example of the value of service, as covered by papers such as this, was recently proved by a preliminary report submitted by the writer. The basis

[21] Cons. Engr., Vancouver, B. C., Canada.

[21a] Received by the Secretary April 10, 1935.

[22] Cons. Engr., Buffalo, N. Y.

[22a] Received by the Secretary April 22, 1935.

of this report was the question: "Can that highway be completed for \$400 000"? The report was wanted (as usual) "at once"—"right away"—"as soon as possible."

This section of highway, less than a mile long, will cross four railroads—under one and over three—requiring one single-span, single-track railroad bridge and one 6-span, highway bridge with a 40-ft roadway. Preliminary plans and estimates had been made by various municipal and railroad organizations over a period of several years up to 1931; item prices and unit quantities spread over a wide range in these former estimates. For a preliminary report such as that required for a case of this kind, the handbooks of a fabricating-erecting steel company will supply weights of steel which agree, with sufficient accuracy, with the weights determined by Fig. 3 of the paper.

This question of "loads" for highway bridges has become of greater importance as the class of traffic has changed. Live loads (moving loads) have increased from an average maximum, say, of 4 tons per unit to an average maximum of 20 tons, or more, per unit; and speeds have increased from 3 to 25 miles per hr, with impact "entering the picture" in a geometrical ratio.

At the beginning of the Twentieth Century, the sale of small bridges identified as "town-bought-bridge-company-competitive" designs, was a "racket" in the State of New York. Structures were sold, not necessarily to the lowest bidder, or to the most reliable firm, but to the most generous payer. As the entire cost must come out of the bridge, the keen competition for work on the part of the bridge companies, and for money on the part of the buyers, created a situation in which the steel in the structure became of minor importance.

Another type of this class of bridge, was the so-called "bedstead design," found principally in the mountainous regions. It was composed of white-painted railings of 1-in. and $1\frac{1}{2}$-in. pipe, and resembled nothing so much as the white iron bedstead found in mountain inns.

The flimsy, spider-web trusses of these bridges are no longer considered "good business" as damage claims are difficult to contest and expensive to settle. The bridge engineer has come into his own on highway bridges of practically all lengths and it is with information such as that supplied in this paper that the engineer who has devoted his time to hydraulics, or some other specialty, may be saved much time in telling his client whether a general plan is possible with the money available. In this manner, subsequently, another case is provided for the bridge expert.

An idea of the present condition of thought upon this matter of loads for highway bridges may be gathered from a study of the following specifications[23]:

(*a*) The highway loading shall be of three classes, namely, *H*-20, *H*-15, and *H*-10, and may be either truck train loadings or equivalent loadings. Loadings *H*-15 and *H*-10 are 75% and 50%, respectively, of Loading *H*-20.

[23] "Standard Specifications for Highway Bridges and Incidental Structures," by the Am. Assoc. of State Highway Officials, 1931; Specifications 5.2.7, p. 174, and 5.2.10, Fig. 5, p. 176.

The truck train loading shall be as shown in Fig. 13 and shall be used for loaded lengths of less than 60 ft. It shall consist of one truck of the gross weight indicated by the loading class followed by, or preceded by (or both followed and preceded by) a line of trucks of indefinite length, each of the following or preceding trucks having a gross weight of three-fourths that indicated by the loading class.

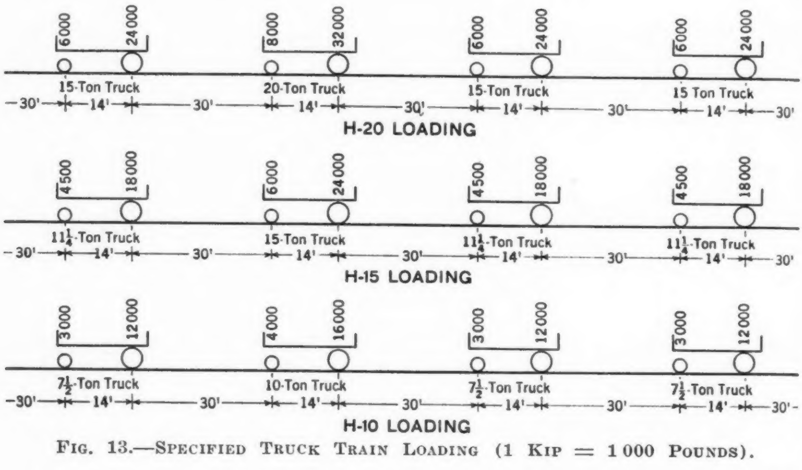

FIG. 13.—SPECIFIED TRUCK TRAIN LOADING (1 KIP = 1 000 POUNDS).

(*b*) If the loaded width of the roadway exceeds 18 ft, the specified loads shall be reduced 1% for each foot of loaded roadway width in excess of 18 ft, with a maximum reduction of 25 per cent.

The *H*-20 loading referred to in Specification (*a*) is shown in Fig. 13. Loadings *H*-15 and *H*-10 differ only in the applied loads, the spacing of trucks being the same for all classes. Specification (*b*) indicates the permissible reduction in "load intensity." If the author has used the *H*-15 and *H*-10 loadings as his Class *A* and Class *B* loadings (which is what they are termed in the Standard Specifications[23]), it is believed that his paper is not consistent with present practice in the Engineering Departments of the State of New York. As far as the writer can learn this State is designing, and approving the design of, bridges under Class *H*-20 loadings and using 20 ft as the distance between trucks.

Highway bridges are now being built with 40-ft roadways, consisting of reinforced concrete pavements 12 in. thick and, where they cross a railroad, the steel floor systems are cement-coated. Under such construction the dead weight of reinforced concrete per linear foot of bridge, is about 3.83 tons; to carry this concrete the floor system will weigh about 0.54 ton per lin ft of bridge. Adding impact loads, wind pressure, or the weights of curbs, sidewalks, snow, pipe lines, etc., indicates a wide departure from the spider-web type of truss.

Loadings for highway bridges present an open question at the present time. Various types of steel floors are being studied in order to reduce the

tremendous weights of solid concrete or block pavement. Live loads are pure assumption based, perhaps, upon the hope that the bridge will endure even if the truck operator insists on transporting all the freight in the country in unregulated loads and at, practically, unregulated speeds.

As a large majority of the bridges being built upon highways (those carrying the obstruction over the highway as well as those carrying the highway over the obstruction) are of 300 ft, or less, in span length, it is hoped that the author will describe somewhat in more detail loadings and other factors that affect Figs. 1 to 5. The final design must always be made by the bridge engineer, as guessing and "skinning" are ruled out when dealing with present-day, and probable future, traffic weights, volumes, and speeds.

T. Kennard Thomson,[24] M. Am. Soc. C. E. (by letter).[24a]—Recalling vividly an experience of more than forty years ago, the writer does not hesitate to state that, had it been available, this paper would have been worth considerable money to him in times past. He had been assigned, suddenly, the task of designing and estimating the cost of a bridge over the East River in New York City, to be composed of six tracks, two roadways, and two sidewalks. The maximum time allotted to the task was one week.

The only truss bridge of comparable length at that time was the Firth of Forth Bridge in Scotland which had been described[25] by the late Sir Benjamin Baker, Hon. M. Am. Soc. C. E. Designing the floor system carefully and the lateral system proportionally, the writer used the dead-load weights of the Forth Bridge in solving the stresses for the bridge graphically. By successive repetitions of this procedure a schedule of dead loads was determined which seemed satisfactory.

Four bridge companies were requested to check the design and the quantities, and the only organization that could be interested required the services of a squad of men for four months to produce the complete design.

The foregoing account demonstrates roughly, the tremendous amount of time that can be saved by the use of curves, such as those in the paper. Even in those days, the designer had at least one standard handbook, produced by steel manufacturers. They had an infinitely easier job than the late James B. Eads, F. Am. Soc. C. E., for example, who had to invent his own structural shapes and fight to have them rolled.

The paper is the result of such tremendous amount of strenuous and careful work, that all bridge designers to whom it is made available should be grateful for its presentation.

[24] Cons. Engr., New York, N. Y.

[24a] Received by the Secretary April 22, 1935.

[25] "Bridging the Firth of Forth," by Sir Benjamin Baker, Lond., 1887.

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DISCUSSIONS

RATIONAL DESIGN OF STEEL COLUMNS

Discussion

BY MARVIN A. GRAY, ESQ.

MARVIN A. GRAY, ESQ.[31] (by letter).[31a]—Long before Samson and Delilah, which is more than 3 000 years ago, columns have played an important part in tragedies. Any study of the structures of ancient times, as far back as records go, shows that an inherent subconscious fear of the combined features of lack of strength and stability in columns existed. This can be observed readily in the pictures of ancient buildings. The ancients built their columns by what they thought was a rational method of design; and yet many workmen were killed in the construction of their larger buildings and bridges. On August 29, 1907, the Quebec Bridge collapsed, one of the world's greatest structural failures; to-day, tragedies of column failure occur even in airplanes, which are more carefully and exactly designed than any other structure built. Therefore, this paper is on a most useful subject, presented at an opportune time; but it touches only lightly on the subject.

With the coming of Galileo there is record of a definite beginning in the development of the structural theory. Then came Hooke and Huygens with their conception of elasticity. In 1696 Jacques Bernoulli[32] proposed the problem[33] which led to Euler's discovery of the column formula; but this did not happen until a clever experimentalist found that column strength varied directly as the area and inversely as the square of the length. In 1744[33] Euler first developed his now famous column formula; in 1757[34] he developed the formula in a more general form and discussed it in great detail. Then came Thomas Young (1773–1829)[35] with his very practical conception of E. In 1855, it remained for Bessemer first to produce an

NOTE.—The paper by D. H. Young, Jun. Am. Soc. C. E., was published in December, 1934, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1935, by Messrs. William R. Osgood, Alfred S. Niles, J. F. Baker, and K. L. DeBlois.

[31] Chicago, Ill.

[31a] Received by the Secretary February 19, 1935.

[32] Encyclopædia Britannica, Fourteenth Edition, Vol. III, p. 457.

[33] Leonard Euler: "Methodus inveniendi Lineas Curvas maximi minimique proprietate gaudentes, sive Solutio problematis isoperimetrici lastissimo sensu accepti." Lausannae & Genevae, MDCCXLIV; Additamentum I, "De Curvis elasticis," pp. 245 to 311.

[34] K. Preussische akademie der wissenschaften Berlin Histoire * * * avec les memoires, 1757. (*Memoires* de l'Academie de Berlin, Tome XIII, 1757, p. 252.)

[35] Dictionary of National Biography, Vol. LXIII, printed in 1900, p. 396 (also, his lectures, i 137).

economical and strong metal building material. As time has passed, many column formulas have been proposed by such men as Euler, LaGrange, Schneider, Rankine, Johnson, Engesser, etc., and special formulas by such men as Gordon, Ritter, Swain, Merriman, Considére, von Kármán, Westergaard, E. H. Salman, O. H. Basquin, Hickerson, D. H. Young, etc.

None of these men has said much more than Euler and LaGrange; and they have omitted and warped much of the original. The formula proposed in the paper is in direct contradiction with those derived by J. Prescott, for example.[36] Too much attention has been paid to producing new formulas and not enough to securing more accurate data and applying them. More tests on really high-strength rolled columns of solid cross-section or rolled structural shape are needed. This would be valuable rational data for a rational column formula.

The Engineers of the U. S. Army Air Corps have derived twelve formulas, illustrated by graphs, which are more accurate and useful than any of the other column formulas (except that by Euler).[37] These formulas involve the use of E as a constant although some steels have a proportional limit of 51 000 lb per sq in. and a yield point of 174 000 lb per sq in.[38] The author also assumes E to be constant although in Fig. 1 it is quite apparent that it may vary considerably from $f = 0$ to $f = f_y$, because then E becomes the secant modulus, and, in many tests, E has been shown to decrease from 60 to 70%, although the material is able to keep a constant load. In tests conducted at the Structural Laboratory of Northwestern University School of Engineering, the writer found that columns reach their maximum strength after they have deflected laterally. This fact was noted in tests recorded many years ago by the late George F. Swain, Past-President and Hon. M. Am. Soc. C. E.,[39] and also in the formula developed by Euler,[33] namely,

$$P = \frac{\pi^2 EI \sqrt{1 + \theta^2}}{L^2} \quad \text{.................... (82)}$$

Mr. Young feels that he should accept such standards as happen to exist. The type that happens to be fabricated most generally is taken as a standard; but what about the sub-average column? To the writer it seems that the standard should be a minimum fixed by engineers rather than by fabricators. He has seen structural sections 10 ft long made as accurate in cross-section as any commercially machined material with an error (or δ) of -0.01 in., or $\frac{\Delta}{L} = \frac{0.01}{120} = \frac{1}{12\,000}$. This is thirty times as accurate as the author's standard. In ordinary work the steel companies can produce an accuracy of $\frac{1}{9\,220}$. From 1874 to 1898 strength of steel produced for the major bridges decreased 40% and from 1898 to 1924 it decreased (with rare exceptions in

[36] "Applied Elasticity," Paragraph 78; see, also, Paragraphs 76 to 93.

[37] "Airplane Design," Revised Edition, February, 1930, U. S. Army Air Corps (see Fig. *A*, p. 350).

[38] Air Corps *Information Circular No. 656*, p. 2, Table 1-A.

[39] "Structural Engineering," by George F. Swain, First Edition, 1924, Vol. I, Chapter on "Columns," p. 446, Lines 26 and 27; p. 448, Line 5 through Line 9.

both cases) $6\frac{1}{2}$ per cent.[40] Nevertheless, for the more important bridges, engineers are consistently thinking of stronger steels, which leads to more and better bridges. Unless useful study of columns is advanced so as to create less expensive bridges (which means also using material of higher specific strength as in airplanes), not many more bridges will be built—although, of necessity, a few will have to be built from time to time. Let the engineer step forward and declare what should, will, and is to be, rather than what may happen to be, taken as the standard.

Little has been concluded, definitely, concerning the strength of steel columns. Every city has its own formula; every State has one, and every group of engineers. Separate column formulas are offered for bridges, buildings, ships, airplanes, and airships. Mr. Young mentions the general fact that there are too many column formulas, but does not demonstrate how he applies his formula to the various types of columns just mentioned. They should be expressed by one formula because they are all fundamentally the same. Investigators, however, always seem to favor some new mathematical expression. In 1757 Euler favored a great number of tests on the subject in question. If one were to translate Euler's work and LaGrange's work completely and undertake a series of useful, complete, but inexpensive column tests on the basis of their findings, one would at least have a rational idea of the problem. In studying built-up columns at present, the profession seems to be complicating the column subject rather than leading the way to a stiffer and stronger column.

Although the author touches on many important points of the subject in question, it is not clear how he will apply his formula to a member of great strength, and as a guide or aid in research. His formula does not follow that of Euler in the range that Euler's has been proved valid, or for columns that are straight. The work of the Society's Special Committee on Steel Column Research[41] presents many points of importance not explained by Mr. Young's paper. The unit stresses used to-day in column design represent about 6% of the unit stresses developed in columns that have been tested. The design of rigid frames is not a complex problem of designing columns, but a combination of the analysis of column and beam action. Elasticity, crookedness, eccentricity, and end conditions of columns have been, and can be, controlled. This paper reveals the results of considerable thought and labor and this discussion is intended only to emphasize the fact that the data upon which it is based are inadequate, when E, an assumed constant, may vary from 60 to 70 per cent.

[40] "Structural Engineering," by George W. Swain, Past-President and Hon. M. Am. Soc. C. E., First Edition, 1924, Vol. II, p. 93.

[41] *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1376.